Homotopy Analysis Method to Solve the Multi-Order Fractional Differential Equations

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Abstract

This paper applies the homotopy analysis method (HAM) to obtain the solution of multi-order fractional differential equation. The fractional derivative is described in Caputo sense. Some test examples have been present.

Index terms — Fractional differential equation, Caputo’s fractional derivative, Riemann-Liouville fractional derivative, homotopy analysis method.

1 I. INTRODUCTION

In recent years, fractional differential equations have attracted many researchers [7,8,9] due to their very important applications in Physics, Science and Engineering such as damping law, rheology, diffusion process, description of fractional random walk and so on. Most fractional differential equations do not have exact solutions, so approximation and numerical techniques must be used, such as Laplace transform method [10], Adomian decomposition method [6,11,12], Variational iteration method [13,14], Homotopy perturbation method [15,16], Hamotopy analysis method [1,17,18] and so on. The homotopy analysis method (HAM) was first proposed by Liao [1] in his Ph.D. Thesis. This method (HAM) given in Liao [17] also provides a systematic and an effective procedure for explicit and numerical solutions of a wide and general class of differential equations system representing real physical and engineering problems.

In this paper, the homotopy analysis method (HAM) Liao [1] is applied to solve multi-order fractional differential equations studied by Diethelm and Ford [2]. We also present an algorithm to convert the multi-order fractional differential equation into a system of fractional differential equations without putting any of the restrictions. This algorithm is valid in the most general case and yields fewer number of equations in a system compared to those in Diethelm-Ford algorithm. In last the solutions of the system of FDE have been obtained by applying the Homotopy analysis method.

2 II.

3 SOME BASIC DEFINITIONS Definition 2.1:

A real function \( F(x) \), \( x > 0 \) is said to be in space is there exists a real number \( p ( > ) \) such that \( F(x) = x^p F_1(x) \) where \( F_1(x) \in [0, \] and it is said to be in the space .

Definition 2.2:

The Riemann-Liouville fractional integral operator of order of a function defined as (1) Properties of the operator \( J^\alpha \) can be found in [10,19] we mentioned only the following (2) Definition 2.3 :

The fractional derivative of \( F(x) \) in the Caputo sense is defined as(3) For (4)

Caputo’s fractional derivative has a useful property [19] (5), R N. m c F iff C (m) m 0 c . F , is \( F(x) F(x) J^0 \) 0 x 0 ; dt F(t) t x 1 F(x) J 0 x 0 (i) J J J J J J J J (ii) J J J J J J (iii) x 1 x J For F c , 1, and 0 c dt t F(t) x n 1 F(x) D J F(x) D (n) 1 n x 0 n n 1 c F 0, x, _n n, n 1 n n x 1 n ! k 0 F F(x) F(x) D J J J k (k) n 0 k I 2012 March

For the Caputo’s derivative (6) [7] Caputo’s fractional derivative is linear operator, similar to integer order derivative (8) where a and b are constants. Also this operator satisfies the so-called Leibnitz rule. (9) For n to be the smallest integer that exceeds , the Caputo space fractional derivative of order \( ? > 0 \) is defined as
For the purpose of this article, the Caputo’s definition of fractional differentiation will be used. Definition 
2.4 : The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined as the following series representation valid in the 
whole complex plane [3]. Lemma 2.5. Diethelm and Ford [4]. Let $Y(t, o, t)$ for some $T > 0$ and and let $q \in \mathbb{N}$ 
be such that $0 < q < k$ then III.

4 ALGORITHM TO CONVERT THE MULTI-ORDER FDE 
INTO A SYSTEM OF FDE 

Let the given fractional differential equation is (11) Subject to the initial conditions (12) where for all $i = 1, 2, \ldots, k$ 
and assume that

In Daftardor-Gejji and Jafari [5], Jafari, Das and Tajadodi [6] it was proved that the FDE (11) can be 
represented as a system of FDE, without any additional restrictions mentioned in equation (12). Here is above 
mentioned approach. Let us define then (13) Here two cases arise Case (i) : If then define (14) 

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Volume XII Issue v [16] and continuing similarly one can convert the initial value problem (11) into a system of FDE. 
and (22) 

Thus as $q$ increases from 0 to 1 the solution varies from the initial guess to the solution. Expanding in Taylor’s 
series with respect to $q$, we have

We can present the multi-order equation (11) as system of fractional differential equations: Obviously, when $q = 0$ and $q = 1$ it holds (22)

IV.

6 BASIC IDEA OF HAM AND A SYSTEM OF FDE 

We have

The following example will illustrate the method. Consider (17) where (18) This initial value problem can be 
viewed as the following system of FDE. (19) This algorithm is valid in the most general case, because we do not 
impose any of the restriction on $\alpha$ and $n$. As mentioned in equation (12).

IV.

7 TEST EXAMPLES

Example 1.

(17) 

with the initial conditions (33) 

In view of the discussion in the last section the equation (32) can be viewed as the following system of FDE 
then and Using equation (1), we get the following scheme: and hence In view of above terms, we find $y_1(x) = 
x^3, y_2(x) = 0$ so $y(x) x$ is the required solution of the given equation.
This paper deals with the approximate solution of a class of multi-order fractional differential equations by Homotopy analysis method. Thus it has been demonstrated that Homotopy analysis method proves useful in solving linear as well as non-linear multi-order fractional differential equation by reducing them into a system of fractional differential equations.

Equation (??4) is equivalent to the following system of equations as the initial guess we assume $y_{10} = 1 + x$, $y_{20} = 0$.

By HAM the $m$-th order deformation equations are given by $y_{1}(x) = y(x) x^{m} + y_{j} x^{j} h$

\[\text{References Références Referencias}\
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