Global Journal of researches in engineering
Numerical Methods
Volume 12 Issue 1 Version 1.0 March 2012
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4596 & Print ISSN: 0975-5861

Quintic B-spline Collocation Method For Sixth Order Boundary Value Problems

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GJRE-I Classification : FOR Code: 010399

QUINTIC B-SPLINE COLLOCATION METHOD FOR SIXTH ORDER BOUNDARY VALUE PROBLEMS

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Quintic B-spline Collocation Method For Sixth Order Boundary Value Problems

K.N.S. Kasi Viswanadham [α] & Y. Showri Raju [σ]

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I. Introduction

In this paper, we developed a collocation method with quintic B-splines as basis functions for getting the numerical solution of a general sixth order linear boundary value problem, which is in the form

$$a_0(x)y^{(6)}(x) + a_1(x)y^{(5)}(x) + a_2(x)y^{(4)}(x)$$
$$+a_3(x)y'''(x) + a_4(x)y''(x) + a_5(x)y'(x)$$
$$+ a_6(x)y(x) = b(x), \qquad c < x < d \qquad (1)$$

subject to the boundary conditions

$$y(c) = A_0, \qquad y(d) = B_0$$
$$y'(c) = A_1, \qquad y'(d) = B_1$$
$$y''(c) = A_2, \qquad y''(d) = B_2. \qquad (2)$$

where A_0 , B_0 , A_1 , B_1 , A_2 , B_2 are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $a_5(x)$, $a_6(x)$ and $b(x)$ are all continuous functions defined on the interval $[c, d]$.

Generally, these types of differential equations have special significance in astrophysics. The dynamo action in some stars may be modeled by sixth order boundary value problems [1].

The narrow convecting layers bounded by stable layers, which are believed to surround A-type stars, may be modeled by sixth-order boundary value problems [2]. Moreover, when an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. When this instability is an ordinary convection, the ordinary differential equation is a sixth-order ordinary differential equation. For further discussion of sixth-order boundary value problems, see [3,4,5].

The existence and uniqueness of solution of such type of boundary value problems can be found in the book written by Agarwal [6]. El-Gamel et al. [7] used Sinc-Galerkin method to solve sixth order boundary value problems. Akram and Siddiqi [8] solved the boundary value problem of type (1)-(2) with non-polynomial spline technique. Siddiqi et al.[9] solved the same boundary value problems using quintic splines. Also Siddiqi and Akram [10] used septic splines to solve the boundary value problems of type (1)-(2). Lamini et al. [11] used spline collocation method to solve the sixth-order boundary value problems.

Further, decomposition methods [12], Ritz's method based on the variational theory [13], and the homotopy perturbation method [14] have been applied for the solution of sixth-order boundary value problems. Variational iteration method for solving sixth-order boundary value problems have been developed by Noor et al.[15]. Siraj-ul-Islam et al. [16,17] developed non-polynomial splines approach to the solution of sixth-order and fourth-order boundary-value problems. Viswanadham et al. [18,19] used sixth order and septic B-splines to solve sixth order boundary value problems.

In this paper, we try to present a simple collocation method using quintic B-splines as basis functions to solve the sixth order boundary value problem of the type (1)-(2).

In section 2 of this paper, the justification for using the collocation method has been mentioned. In section 3, the definition of quintic B-splines has been described. In section 4, description of the collocation method with quintic B-splines as basis functions has been presented and in section 5, solution procedure to find the nodal parameters is presented. In section 6, numerical examples of both linear and non-linear boundary value problems are presented. The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization

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Global Journal of Researches in Engineering (I) Volume XII Issue I Version I March 2012

technique [20]. Finally, the last section is dealt with conclusions of the paper.

II. Justification for Using Collocation Method

In finite element method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods such as Ritz's approach, Galerkin's approach, least squares method and collocation method etc. The collocation method seeks an approximate solution by requiring the residual of the differential equation to be identically zero at N selected points in the given space variable domain where N is the number of basis functions in the basis [21]. That means, to get an accurate solution by the collocation method, one needs a set of basis functions which in number match with the number of collocation points selected in the given space variable domain. Further, the collocation method is the easiest to implement among the variational methods of FEM. When a differential equation is approximated by m^{th} order B-splines, it yields $(m+1)^{th}$ order accurate results [22]. Hence this motivated us to use the collocation method to solve a sixth order boundary value problem of type (1)-(2) with quintic B-splines.

III. Definition of Quintic B-Splines

The cubic B-splines are defined in [23,24]. In a similar analogue, the existence of the quintic spline interpolate $s(x)$ to a function in a closed interval $[c,d]$ for spaced knots (need not be evenly spaced)

$$c = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = d$$

is established by constructing it. The construction of $s(x)$ is done with the help of the quintic B-Splines. Introduce ten additional knots $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ and x_{n+5} such that

$$x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$$

$$\text{and} \quad x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5}.$$

Now the quintic B-splines $B_i(x)$ are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-3}^{i+3} \frac{(x_r - x)_+^5}{\pi'(x_r)}, & x \epsilon [x_{i-3}, x_{i+3}] \\ 0, & \text{otherwise} \end{cases}$$

where

$$(x_r - x)_+^5 = \begin{cases} (x_r - x)^5, & \text{if } x_r \geq x \\ 0, & \text{if } x_r \leq x \end{cases}$$

and

$$\pi(x) = \prod_{r=i-3}^{i+3} (x - x_r).$$

It can be shown that the set $\{B_{-2}(x), B_{-1}(x), B_0(x), \cdots, B_n(x), B_{n+1}(x), B_{n+2}(x)\}$ forms a basis for the space $S_5(\pi)$ of quintic polynomial splines. The quintic B-splines are the unique nonzero splines of smallest compact support with knots at

$$x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 < \cdots$$
$$< x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5}$$

IV. Description of the Method

To solve the boundary value problem (1)-(2) by the collocation method with quintic B- splines as basis functions, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+2} \alpha_j B_j(x) \tag{3}$$

where α_j's are the nodal parameters to be determined. In the present method, the internal mesh points are selected as the collocation points. In collocation method, the number of basis functions in the approximation should match with the number of selected collocation points [21]. Here the number of basis functions in the approximation (3) is n+5, where as the number of selected collocation points is n-1. So, there is a need to redefine the basis functions into a new set of basis functions which in number match with the number of selected collocation points. The procedure for redefining the basis functions is as follows:

Using the quintic B-splines described in section 3 and the Dirichlet boundary conditions of (2), we get the approximate solution at the boundary points as

$$y(c) = y(x_0) = \sum_{j=-2}^{2} \alpha_j B_j(x_0) = A_0 \tag{4}$$

$$y(d) = y(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j(x_n) = B_0. \tag{5}$$

Eliminating α_{-2} and α_{n+2} from the equations (3),(4), and (5) we get the approximation for $y(x)$ as

$$y(x) = w_1(x) + \sum_{j=-1}^{n+1} \alpha_j P_j(x) \tag{6}$$

where

$$w_1(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{B_0}{B_{n+2}(x_n)} B_{n+2}(x)$$

and $P_j(x) =$

$$\begin{cases} B_j(x) - \frac{B_j(x_0)}{B_{-2}(x_0)} B_{-2}(x), & \text{for} \quad j = -1,0,1,2 \\ B_j(x), & \text{for} \quad j = 3,4,\ldots,n-3 \\ B_j(x) - \frac{B_j(x_n)}{B_{n+2}(x_n)} B_{n+2}(x), & \text{for} \quad j = n-2, n-1, n, n+1 \end{cases}$$

Global Journal of Researches in Engineering (I) Volume XII Issue I Version I March 2012

Using the Neumann boundary conditions of (2) to the approximate solution $y(x)$ in (6), we get

$$y'(c) = y'(x_0) = w_1'(x_0) + \alpha_{-1}P_{-1}'(x_0) + \alpha_0 P_0'(x_0) + \alpha_1 P_1'(x_0) + \alpha_2 P_2'(x_0) = A_1 \quad (7)$$

$$y'(d) = y'(x_n) = w_1'(x_n) + \alpha_{n-2}P_{n-2}'(x_n) + \alpha_{n-1}P_{n-1}'(x_n) + \alpha_n P_n'(x_n) + \alpha_{n+1}P_{n+1}'(x_n) = B_1. \quad (8)$$

Now, eliminating α_{-1} and α_{n+1} from the equations (6),(7) ,and (8), we get the approximation for $y(x)$ as

$$y(x) = w_2(x) + \sum_{j=0}^{n} \alpha_j Q_j(x) \quad (9)$$

where

$$w_2(x) = w_1(x) + \frac{A_1 - w_1'(x_0)}{P_{-1}'(x_0)} P_{-1}(x) + \frac{B_1 - w_1'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x)$$

and $Q_j(x)$ =

$$\begin{cases} P_j(x) - \dfrac{P_j'(x_0)}{P_{-1}'(x_0)} P_{-1}(x), & \text{for } j = 0,1,2 \\ P_j(x), & \text{for } j = 3,4,\ldots,n-3 \\ P_j(x) - \dfrac{P_j'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x), & \text{for } j = n-2,n-1,n. \end{cases}$$

Using the boundary conditions $y''(c) = A_2$ and $y''(d) = B_2$ of (20) to the approximate solution $y(x)$ in (9), we get

$$y''(c) = y''(x_0) = w_2''(x_0) + \alpha_0 Q_0''(x_0) + \alpha_1 Q_1''(x_0) + \alpha_2 Q_2''(x_0) = A_2 \quad (10)$$

$$y''(d) = y''(x_n) = w_2''(x_n) + \alpha_{n-2}Q_{n-2}'' + \alpha_{n-1}Q_{n-1}''(x_n) + \alpha_n Q_n''(x_n) = B_2. \quad (11)$$

Now, eliminating α_0 and α_n from the equations (9),(10), and (11) we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=1}^{n-1} \alpha_j \tilde{B}_j(x) \quad (12)$$

where

$$w(x) = w_2(x) + \frac{A_2 - w_2''(x_0)}{Q_0''(x_0)} Q_0(x) + \frac{B_2 - w_2''(x_n)}{Q_n''(x_n)} Q_n(x)$$

and $\tilde{B}_j(x)$

$$\begin{cases} Q_j(x) - \dfrac{Q_j''(x_0)}{Q_0''(x_0)} Q_0(x), & \text{for } j = 1,2 \\ Q_j(x), & \text{for } j = 3,4,\ldots,n-3 \\ Q_j(x) - \dfrac{Q_j''(x_n)}{Q_n''(x_n)} Q_n(x), & \text{for } j = n-2,n-1, \end{cases}$$

Now the new basis functions for the approximation $y(x)$ are $\{\tilde{B}_j(x), j = 1,2,\ldots,n-1\}$ and they are in number matching with the number of selected collocation points. Since the approximation for $y(x)$ in (12) is a quintic approximation, let us approximate $y^{(5)}$ and $y^{(6)}$ at the selected collocation points with central differences as

$$y_i^{(5)} = \frac{y_{i+1}^{(4)} - y_{i-1}^{(4)}}{2h} \text{ and } y_i^{(6)} = \frac{y_{i+1}^{(4)} - 2y_i^{(4)} + y_{i-1}^{(4)}}{h^2} \quad (13)$$

where

$$y_i = y(x_i) = w(x_i) + \sum_{j=1}^{n-1} \alpha_j \tilde{B}_j(x_i) \quad (14)$$

Now applying collocation method to (1), we get

$$a_0(x_i)y_i^{(6)} + a_1(x_i)y_i^{(5)} + a_2(x_i)y_i^{(4)} + a_3(x_i)y_i''' + a_4(x_i)y_i'' + a_5(x_i)y_i' + a_6(x_i)y_i = b(x_i)$$
$$\text{for } i=1,2,\ldots,n-1. \quad (15)$$

Using (13) and (14) in (15), we get

$$\frac{a_0(x_i)}{h^2}\Big[w^{(4)}(x_{i+1}) + \sum_{j=1}^{n-1}\alpha_j \tilde{B}_j^{(4)}(x_{i+1}) - 2\Big\{w^{(4)}(x_i) + \sum_{j=1}^{n-1}\alpha_j\tilde{B}_j^{(4)}(x_i)\Big\} + w^{(4)}(x_{i-1}) + \sum_{j=1}^{n-1}\alpha_j\tilde{B}_j^{(4)}(x_{i-1}) \Big]$$
$$+ \frac{a_1(x_i)}{2h}\Big[w^{(4)}(x_{i+1}) + \sum_{j=1}^{n-1}\alpha_j\tilde{B}_j^{(4)}(x_{i+1}) - \Big\{w^{(4)}(x_{i-1}) + \sum_{j=1}^{n-1}\alpha_j\tilde{B}_j^{(4)}(x_{i-1})\Big\} \Big]$$
$$+ a_2(x_i)\Big[w^{(4)}(x_i) + \sum_{j=1}^{n-1}\alpha_j\tilde{B}_j^{(4)}(x_i)\Big]$$
$$+ a_3(x_i)\Big[w'''(x_i) + \sum_{j=1}^{n-1}\alpha_j\tilde{B}_j'''(x_i)\Big]$$
$$+ a_4(x_i)\Big[w''(x_i) + \sum_{j=1}^{n-1}\alpha_j\tilde{B}_j''(x_i)\Big]$$
$$+ a_5(x_i)\Big[w'(x_i) + \sum_{j=1}^{n-1}\alpha_j\tilde{B}_j'(x_i)\Big]$$
$$+ a_6(x_i)\Big[w(x_i) + \sum_{j=1}^{n-1}\alpha_j\tilde{B}_j(x_i)\Big]$$

$$= b(x_i) \qquad \text{for } i=1,2,\ldots,n-1. \qquad (16)$$

Rearranging the terms and writing the system of equations (16) in matrix form, we get

$$A\alpha = B \qquad (17)$$

where

$$A = [a_{ij}];$$

$$a_{ij} = \widetilde{B}_j^{(4)}(x_{i-1})\left(\frac{a_0(x_i)}{h^2} - \frac{a_1(x_i)}{2h}\right)$$
$$+\widetilde{B}_j^{(4)}(x_i)\left(-2\frac{a_0(x_i)}{h^2}\right)$$
$$+\widetilde{B}_j^{(4)}(x_{i+1})\left(\frac{a_0(x_i)}{h^2} + \frac{a_1(x_i)}{2h}\right)$$
$$+\widetilde{B}_j^{(4)}(x_i)a_2(x_i)$$
$$+\widetilde{B}_j'''(x_i)a_3(x_i) + \widetilde{B}_j''(x_i)a_4(x_i)$$
$$+\widetilde{B}_j'(x_i)a_5(x_i) + \widetilde{B}_j(x_i)a_6(x_i)$$

$$\text{for } i=1,2,\ldots,n-1, \quad j=1,2,\ldots,n-1. \qquad (18)$$

$$B = [b_i];$$

$$b_i = b(x_i) - \Big[\; w^{(4)}(x_{i-1})\left(\frac{a_0(x_i)}{h^2} - \frac{a_1(x_i)}{2h}\right)$$
$$+w^{(4)}(x_i)\left(-2\frac{a_0(x_i)}{h^2}\right)$$
$$+w^{(4)}(x_{i+1})\left(\frac{a_0(x_i)}{h^2} + \frac{a_1(x_i)}{2h}\right)$$
$$+w^{(4)}(x_i)a_2(x_i) + w'''(x_i)a_3(x_i)$$
$$+w''(x_i)a_4(x_i) + w'(x_i)a_5(x_i)$$
$$+w(x_i)a_6(x_i) \;\Big]$$

$$\text{for } i=1,2,\ldots,n-1. \qquad (19)$$

and

$$\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]^T.$$

V. Solution Procedure to Find the Nodal Parameters

The basis function $\widetilde{B}_i(x)$ is defined only in the interval $[x_{i-3}, x_{i+3}]$ and outside of this interval it is zero. Also at the end points of the interval $[x_{i-3}, x_{i+3}]$ the basis function $\widetilde{B}_i(x)$ vanishes. Therefore, $\widetilde{B}_i(x)$ is having non-vanishing values at the mesh points $x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and zero at the other mesh points. The first four derivatives of $\widetilde{B}_i(x)$ also have the same nature at the mesh points as in the case of $\widetilde{B}_i(x)$. Using these facts, we can say that the matrix A defined in (18) is a seven diagonal band matrix. Therefore, the system of equations (17) is a seven diagonal band system in $\alpha_i's$. The nodal parameters $\alpha_i's$ can be obtained by using band matrix solution package. We have used the FORTRAN-90 programming to solve the boundary value problem (1)-(2) by the proposed method.

VI. Numerical Examples

To demonstrate the applicability of the proposed method for solving the sixth order boundary value problems of type (1)-(2) we considered seven examples of which four are linear and three are non linear boundary value problems. Numerical results for each problem are presented in tabular forms and compared with the exact solutions or numerical solutions available in the literature.

Example 1 : Consider the linear boundary value problem

$$y^{(6)} + xy =$$
$$-(24 + 11x + x^3)e^x, \quad 0 < x < 1 \qquad (20)$$

subject to

$$y(0) = 0, \; y(1) = 0,$$
$$y'(0) = 1, \; y'(1) = -e, \qquad (21)$$
$$y''(0) = 0, \; y''(1) = -4e.$$

The exact solution for the above problem is given by $y(x) = x(1-x)e^x$. The proposed method is tested on this problem where the domain [0,1] is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 1. The maximum absolute error obtained by the proposed method is 7.361174×10^{-6}

Table 1 : Numerical results for Example 1

x	Exact Solution	Absolute error by proposed method
0.1	9.946539E-02	4.619360E-07
0.2	1.954244E-01	1.847744E-06
0.3	2.834704E-01	3.874302E-06
0.4	3.580379E-01	6.288290E-06
0.5	4.121803E-01	7.361174E-06
0.6	4.373085E-01	7.241964E-06
0.7	4.228881E-01	6.496906E-06
0.8	3.560865E-01	4.649162E-06
0.9	2.213642E-01	2.324581E-06

Example 2: Consider the linear boundary value problem

$$y_{(6)} + e^{-x}y =$$
$$-720 + (x - x^2)^3 e^{-x}, \; 0<x<1 \qquad (22)$$

subject to

$$y(0) = 0,\ y(1) = 0,$$
$$y'(0) = 0,\ y'(1) = 0,$$
$$y''(0) = y''(1) = 0 \qquad (23)$$

The exact solution for the above problem is given by $y(x) = x^3(1-x)^3$. The proposed method is tested on this problem where the domain [0,1] is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 2. The maximum absolute error obtained by the proposed method is $2.942979\ \times 10^{-6}$.

Table 2 : Numerical results for Example 2

x	Exact solution	Absolute error by proposed method
0.1	7.290000E-04	3.894675E-07
0.2	4.096000E-03	1.216307E-06
0.3	9.261001E-03	2.081506E-06
0.4	1.382400E-02	2.698973E-06
0.5	1.562500E-02	2.942979E-06
0.6	1.382400E-02	2.727844E-06
0.7	9.261001E-03	2.096407E-06
0.8	4.096000E-03	1.219101E-06
0.9	7.289993E-04	3.807945E-07

Example 3 : Consider the linear boundary value problem

$$y^{(6)} + y''' + y'' - y =$$
$$e^{-x}(-15x^2 + 78x - 114), \quad 0 < x < 1 \qquad (24)$$

subject to

$$y(0) = 0,\ \ y(1) = \frac{1}{e},$$
$$y'(0) = 0,\ \ y'(1) = \frac{2}{e}, \qquad (25)$$
$$y(0) = 0,\ \ y(1) = \frac{1}{e},$$

The exact solution for the above problem is given by $y(x) = x^3e^{-x}$. The proposed method is tested on this problem where the domain [0,1] is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 3. The maximum absolute error obtained by the proposed method is $4.939735\ \times 10^{-6}$

Table 3 : Numerical results for Example 3

x	Exact solution	Absolute error by proposed method
0.1	9.048374E-04	1.808512E-07
0.2	6.549846E-03	9.508803E-07
0.3	2.000210E-02	2.233312E-06
0.4	4.290048E-02	3.498048E-06
0.5	7.581633E-02	4.529953E-06
0.6	1.185433E-01	4.939735E-06
0.7	1.703288E-01	4.217029E-06
0.8	2.300564E-01	2.712011E-06
0.9	2.963893E-01	1.430511E-06

Example 4 : Consider the linear boundary value problem

$$y^{(6)} + y = 6(2x\cos x + 5\sin x), \qquad -1 < x < 1 \qquad (26)$$

subject to

$$y(-1) = 0,\ \ y(1) = 0,$$
$$y'(-1) = 2\sin 1,\ \ y'(1) = 2\sin 1,$$
$$y''(-1) = -4\cos 1 - 2\sin 1, \qquad (27)$$
$$y''(1) = 4\cos 1 + 2\sin 1.$$

The exact solution for the above problem is given by $y(x) = (x^2-1)\sin x$. The proposed method is tested on this problem where the domain [-1,1] is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 4. The maximum absolute error obtained by the proposed method is 8.076429×10^{-6}.

Table 4 : Numerical results for Example 4

x	Exact solution	Absolute error by proposed method
-0.8	2.582482E-01	4.172325E-07
-0.6	3.613712E-01	1.251698E-06
-0.4	3.271114E-01	1.788139E-07
-0.2	1.907225E-01	2.488494E-06
0.0	0.000000	5.105962E-06
0.2	-1.907226E-01	8.076429E-06
0.4	-3.271114E-01	7.987022E-06
0.6	-3.613712E-01	6.377697E-06
0.8	-2.582482E-01	3.904104E-06

Example 5 : Consider the nonlinear boundary value problem

$$y^{(6)} - 20e^{-36y(x)} =$$
$$-40(1+x)^{-6}, \qquad 0 < x < 1 \qquad (28)$$

subject to

$$y(0) = 0,\ \ y(1) = \frac{1}{6}\ln 2,$$
$$y'(0) = \frac{1}{6},\ \ y'(1) = \frac{1}{12}, \qquad (29)$$
$$y''(0) = \frac{-1}{6},\ \ y''(1) = \frac{-1}{24}.$$

The exact solution for the above problem is given by $y(x) = \frac{1}{6}\ln(1+x)$. This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique [20] as

$$y^{(6)}_{(n+1)} + \left[720e^{-36y_{(n)}}\right]y_{(n+1)} =$$
$$720y_{(n)}e^{-36y_{(n)}} + 20e^{-36y_{(n)}}$$

$$-40(1+x)^{-6}, \quad \text{for} \quad n=0,1,2,\ldots \tag{30}$$

subject to

$$y_{(n+1)}(0)=0,\ y_{(n+1)}(1)=\frac{1}{6}\ln 2,$$
$$y'_{(n+1)}(0)=\tfrac{1}{6},\ y'_{(n+1)}(1)=\tfrac{1}{12}, \tag{31}$$
$$y''_{(n+1)}(0)=\frac{-1}{6},\ y''_{(n+1)}(1)=\frac{-1}{24}.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for y. The domain [0,1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of problems (30). Numerical results for this problem are presented in Table 5. The maximum absolute error obtained by the proposed method is 1.329929×10^{-6}.

Table 5 : Numerical results for Example 5

x	Exact solution	Absolute error by proposed method
0.1	1.588503E-02	1.303852E-07
0.2	3.038693E-02	5.215406E-07
0.3	4.372738E-02	9.723008E-07
0.4	5.607871E-02	1.329929E-06
0.5	6.757752E-02	1.259148E-06
0.6	7.833394E-02	8.419156E-07
0.7	8.843804E-02	4.023314E-07
0.8	9.796445E-02	8.195639E-08
0.9	1.069757E-01	1.713634E-07

Example 6 : Consider the nonlinear boundary value problem

$$y^{(6)}+e^{-x}y^2=e^{-x}+e^{-3x}, \quad 0<x<1 \tag{32}$$

subject to

$$y(0)=1,\ y(1)=\frac{1}{e},$$
$$y'(0)=-1,\ y'(1)=\frac{-1}{e} \tag{33}$$
$$y''(0)=1,\ y''(1)=\frac{1}{e}.$$

The exact solution for the above problem is given by $y(x)=e^{-x}$. This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique [20] as

$$y^{(6)}_{(n+1)}+[2e^{-x}y_{(n)}]y_{(n+1)}=$$
$$e^{-x}y^2{}_{(n)}+e^{-x}+e^{-3x}, \quad \text{for} \quad n=0,1,2,\ldots \tag{34}$$

subject to

$$y_{(n+1)}(0)=1,\ y_{(n+1)}(1)=\frac{1}{e},$$
$$y'_{(n+1)}(0)=-1,\ y'_{(n+1)}(1)=\frac{-1}{e}, \tag{35}$$
$$y''_{(n+1)}(0)=1,\ y''_{(n+1)}(1)=\frac{1}{e}.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for y. The domain [0,1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of problems (34). Numerical results for this problem are presented in Table 6. The maximum absolute error obtained by the proposed method is 1.913309×10^{-5}.

Table 6 : Numerical results for Example 6

x	Exact solution	Absolute error by proposed method
0.1	9.048374E-01	1.907349E-06
0.2	8.187308E-01	7.033348E-06
0.3	7.408182E-01	1.347065E-05
0.4	6.703200E-01	1.877546E-05
0.5	6.065307E-01	1.913309E-05
0.6	5.488116E-01	1.478195E-05
0.7	4.965853E-01	8.702278E-06
0.8	4.493290E-01	3.188848E-06
0.9	4.065697E-01	2.980232E-08

Example 7 : Consider the nonlinear boundary value problem

$$y^{(6)}=e^{x}y^3(x), \quad 0<x<1 \tag{36}$$

subject to

$$y(0)=1,\ y(1)=e^{\frac{-1}{2}},$$
$$y'(0)=\frac{-1}{2},\ y'(1)=\frac{-1}{2}e^{\frac{-1}{2}}, \tag{37}$$
$$y''(0)=\frac{1}{4},\ y''(1)=\frac{1}{4}e^{\frac{-1}{2}}.$$

This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique [20] as

$$y^{(6)}_{(n+1)}+[-3e^{x}y^2_{(n)}]y_{(n+1)}=$$
$$-2e^{x}y^3_{(n)} \quad \text{for} \quad n=0,1,2,\ldots \tag{38}$$

subject to

$$y_{(n+1)}(0)=1,\ y_{(n+1)}(1)=e^{\frac{-1}{2}},$$

$$y'_{(n+1)}(0) = \frac{-1}{2},\ \ y'_{(n+1)}(1) = \frac{-1}{2}e^{\frac{-1}{2}}, \tag{39}$$

$$y''_{(n+1)}(0) = \frac{1}{4},\ \ y''_{(n+1)}(1) = \frac{1}{4}e^{\frac{-1}{2}}.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for y. The domain [0,1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of problems [38]. The exact solution for the problem (36) and (37) is not available in the literature. The numerical solutions can be obtained for this problem by refining the mesh size. Hussin and Kilicman [25] obtained the numerical solutions for this problem by refining the mesh size. Numerical results obtained by the proposed method are compared with the numerical results obtained by Hussin and Kilicman [25], and the results are presented in Table 7.

Table 7 : Numerical results for Example 7

x	Numerical solutions obtained by Hussin and Kilicman [25]	Absolute error by the proposed method when compared with Hussin and Kilicman [25]
0.1	9.512295E-01	2.563000E-06
0.2	9.048374E-01	1.049042E-05
0.3	8.607080E-01	2.151728E-05
0.4	8.187308E-01	3.141165E-05
0.5	7.788008E-01	3.391504E-05
0.6	7.408182E-01	2.819300E-05
0.7	7.046881E-01	1.811981E-05
0.8	6.703200E-01	7.688999E-06
0.9	6.376281E-01	5.960464E-07

VII. Conclusion

In this paper, we have developed a collocation method with quintic B-splines as basis functions to solve sixth order boundary value problems. Here we have taken internal mesh points as the selected collocation points. The quintic B-spline basis set has been redefined into a new set of basis functions which in number match with the number of selected collocation points. The proposed method is applied to solve several number of linear and non-linear problems to test the efficiency of the method. The numerical results obtained by the proposed method are in good agreement with the exact solutions or numerical solutions available in the literature. The objective of this paper is to present a simple method to solve a sixth order boundary value problem and its easiness for implementation.

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