

Quintic B-spline Collocation Method For Sixth Order Boundary Value Problems

K.N.S.Kasi Viswanadham¹ and Y.Showri Raju²

¹ National Institute of Technology, Warangal (INDIA)zgg

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Abstract

A finite element method involving collocation method with quintic B-splines as basis functions have been developed to solve sixth order boundary value problems. The sixth order and fifth order derivatives for the dependent variable are approximated by the central differences of fourth order derivatives. The basis functions are redefined into a new set of basis functions which in number match with the number of collocated points selected in the space variable domain. The proposed method is tested on several linear and non-linear boundary value problems. The solution of a non-linear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique. Numerical results obtained by the present method are in good agreement with the exact solutions or numerical solutions available in the literature.

Index terms— collocation method, quintic B-spline, basis function, sixth order boundary value problem, absolute error.

1 INTRODUCTION

In this paper, we developed a collocation method with quintic B-splines as basis functions for getting the numerical solution of a general sixth order linear boundary value problem, which is in the form $y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y^{(2)} + a_1 y' + a_0 y = f(x)$, $0 < x < 1$ subject to the boundary conditions $y(0) = y(1) = y'(0) = y'(1) = 0$, where $A_0, B_0, A_1, B_1, A_2, B_2$ are finite real constants and $f_0(x), f_1(x), f_2(x), f_3(x), f_4(x), f_5(x)$ and $f_6(x)$ are all continuous functions defined on the interval $[0, 1]$.

Generally, these types of differential equations have special significance in astrophysics. The dynamo action in some stars may be modeled by sixth order boundary value problems [1].

The narrow convecting layers bounded by stable layers, which are believed to surround A-type stars, may be modeled by sixth-order boundary value problems [2]. Moreover, when an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. When this instability is an ordinary convection, the ordinary differential equation is a sixth-order ordinary differential equation. For further discussion of sixth-order boundary value problems, see [3,4,5].

The existence and uniqueness of solution of such type of boundary value problems can be found in the book written by Agarwal [6]. El-Gamel et al. [7] used Sinc-Galerkin method to solve sixth order boundary value problems. Akram and Siddiqi [8] solved the boundary value problem of type (1)-(2) with nonpolynomial spline technique. Siddiqi et al. [9] solved the same boundary value problems using quintic splines. Also Siddiqi and Akram [10] used septic splines to solve the boundary value problems of type (1)-(2). Lamini et al. [11] used spline collocation method to solve the sixth-order boundary value problems.

Further, decomposition methods [12], Ritz's method based on the variational theory [13], and the homotopy perturbation method [14] have been applied for the solution of sixth-order boundary value problems. Variational

iteration method for solving sixth-order boundary value problems have been developed by Noor et al. [15]. Siraj-ul-Islam et al. [16,17] developed nonpolynomial splines approach to the solution of sixth order and fourth-order boundary-value problems. Viswanadham et al. [18,19] used sixth order and septic B-splines to solve sixth order boundary value problems.

In this paper, we try to present a simple collocation method using quintic B-splines as basis functions to solve the sixth order boundary value problem of the type (1)-(2).

In section 2 of this paper, the justification for using the collocation method has been mentioned. In section 3, the definition of quintic B-splines has beenI Global Journal of Researches in Engineering Volume XII Issue v v v I Version I 1 (D D D D)

described. In section 4, description of the collocation method with quintic B-splines as basis functions has been presented and in section 5, solution procedure to find the nodal parameters is presented. In section 6, numerical examples of both linear and non-linear boundary value problems are presented. The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique [20]. Finally, the last section is dealt with conclusions of the paper.

2 II.

3 JUSTIFICATION FOR USING COLLOCATION METHOD

In finite element method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods such as Ritz's approach, Galerkin's approach, least squares method and collocation method etc. The collocation method seeks an approximate solution by requiring the residual of the differential equation to be identically zero at N selected points in the given space variable domain where N is the number of basis functions in the basis [21]. That means, to get an accurate solution by the collocation method, one needs a set of basis functions which in number match with the number of collocation points selected in the given space variable domain. Further, the collocation method is the easiest to implement among the variational methods of FEM. When a differential equation is approximated by ?? order B-splines, it yields $(n + 1)$ order accurate results [22]. Hence this motivated us to use the collocation method to solve a sixth order boundary value problem of type (1)-(2) with quintic B-splines.

4 III. DEFINITION OF QUINTIC B-SPLINES

The cubic B-splines are defined in [23,24]. In a similar analogue, the existence of the quintic spline interpolate $??(??)$ to a function in a closed interval $[a, b]$ for spaced knots (need not be evenly spaced) $a = x_0 < x_1 < x_2 < \dots < x_n = b$ is established by constructing it. The construction of $??(??)$ is done with the help of the quintic B-Splines.

5 Introduce

$??_5 < ??_4 < ??_3 < ??_2 < ??_1 < ??_0 < a < x < b < ??_1 < ??_2 < ??_3 < ??_4 < ??_5$
 $< ??_5 + 5$
 IV.

6 DESCRIPTION OF THE METHOD

To solve the boundary value problem (1)-(2) by the collocation method with quintic B-splines as basis functions, we define the approximation for $??(??)$ as $??(??) = \sum_{j=1}^{n+1} ??_j B_j(??) = \sum_{j=1}^{n+1} ??_j B_j(??)$ (3)

where $??_j$ are the nodal parameters to be determined. In the present method, the internal mesh points are selected as the collocation points. In collocation method, the number of basis functions in the approximation should match with the number of selected collocation points [21]. Here the number of basis functions in the approximation (3) is $n+5$, where as the number of selected collocation points is $n-1$. So, there is a need to redefine the basis functions into a new set of basis functions which in number match with the number of selected collocation points. The procedure for redefining the basis functions is as follows:

Using the quintic B-splines described in section 3 and the Dirichlet boundary conditions of (2), we get the approximate solution at the boundary points as $??(a) = ??(x_0) = \sum_{j=1}^{n+1} ??_j B_j(a) = \sum_{j=1}^{n+1} ??_j B_j(a) = 0$ (4) $??(b) = ??(x_n) = \sum_{j=1}^{n+1} ??_j B_j(b) = \sum_{j=1}^{n+1} ??_j B_j(b) = 0$ (5)

Eliminating $??_2$ and $??_{n+2}$ from the equations (3), (4), and (5) we get the approximation for $??(??)$ as where $??_1, ??_3, \dots, ??_n$ are the nodal parameters to be determined. In the present method, the internal mesh points are selected as the collocation points. In collocation method, the number of basis functions in the approximation should match with the number of selected collocation points [21]. Here the number of basis functions in the approximation (3) is $n+5$, where as the number of selected collocation points is $n-1$. So, there is a need to redefine the basis functions into a new set of basis functions which in number match with the number of selected collocation points. The procedure for redefining the basis functions is as follows:

Using the Neumann boundary conditions of (2) to the approximate solution $??(??)$ in (6), we get $??'(a) = ??'(x_0) = \sum_{j=1}^{n+1} ??_j B_j'(a) = \sum_{j=1}^{n+1} ??_j B_j'(a) = 0$ and $??'(b) = ??'(x_n) = \sum_{j=1}^{n+1} ??_j B_j'(b) = \sum_{j=1}^{n+1} ??_j B_j'(b) = 0$

$(?? 0) = ?? 1 (7) ?? ? (??) = ?? ? (?? ??) = ?? 1 ? (?? ??) + ?? ???2 ?? ???2 ? (?? ??) + ?? ???1 ??$
 $???1 ? (?? ??) + ?? ?? ?? ?? ? (?? ??) + ?? ??+1 ?? ?? +1 ? (?? ??) = ?? 1 . (8)$
 Now, eliminating $?? ?1$ and $?? ??+1$ from the equations (6), (7), and (8), we get the approximation for $??(??)$
 $as ??(??) = ?? 2 (??) + ? ?? ?? ?? ?? (??) ?? ?? = 0(9)$
 where and $Q_j(x) =$
 Using the boundary conditions $?? ?? (??) = ?? 2$ and $?? ?? (??) = ?? 2$ of (20) to the approximate solution
 $??(??)$ in (9), we get $?? ?? (??) = ?? ?? (?? 0) = ?? 2 ?? (?? 0) + ?? 0 ?? 0 ?? (?? 0) + ?? 1 ?? 1 ?? (?? 0)$
 $+ ?? 2 ?? 2 ?? (?? 0) = ?? 2(10) ?? ?? (??) = ?? ?? (?? ??) = ?? 2 ?? (?? ??) + ?? ???2 ?? ???2 ?? + ??$
 $???1 ?? ???1 ?? (?? ??) + ?? ?? ?? ?? ?? (?? ??) = ?? 2 . (11)$
 Now, eliminating $?? 0$ and $?? ??$ from the equations (9), (10), and (11) we get the approximation for $??(??)$
 $as ??(??) = ??(??) + ? ?? ?? ?? ?? (??) ???1 ?? = 1(12)$
 where $??(??) = +$ and
 Now the new basis functions for the approximation $y(x)$ are $?B_j ?(x)$, $j = 1, 2, ? , n ? 1$ and they are
 in number matching with the number of selected collocation points. Since the approximation for $y(x)$ in (12)
 is a quintic approximation, let us approximate $y(5)$ and $y(6)$ at the selected collocation points with central
 differences $ash y y y i i i 2) 4 (1) 4 (1) 5 (? + ? = and 2) 4 (1) 4 (1) 6 (2 h y y y y i i i i ? + + ?$
 $= (13)$
 where $y_i = y(x_i) = w(x_i) + ? ? j B_j ?(x_i) n ? 1 j = 1(14)$
 Now applying collocation method to (1), we get $?? 0 (?? ??) ?? ?? (6) + ?? 1 (?? ??) ?? ?? (5) + ?? 2 (??$
 $??) ?? ?? (4)$
 Using (??3) and (14) in (15), we get $?? 0 (?? ??) ? 2 [w(4)(x_{i+1}) + ? ? j B_j ?(4)(x_{i+1}) n ? 1 j = 1 ? 2$
 $?w(4)(x_i) + ? ? j B_j ?(4)(x_i) n ? 1 j = 1 ? + w(4)(x_{i?1}) + ? ? j B_j ?(4)(x_{i?1}) n ? 1 j = 1]$ Quintic B-
 $a_1(x_i) 2h [w(4)(x_{i+1}) + ? ? j B_j ?(4)(x_{i+1}) n ? 1 j = 1 ? ?w(4)(x_{i?1}) + ? ? j B_j ?(4)(x_{i?1}) n ? 1$
 $j = 1 ?] + a_2(x_i) ?w(4)(x_i) + ? ? j B_j ?(4)(x_i) n ? 1 j = 1 ? + a_3(x_i) ?w ??? (x_i) + ? ? j B_j ?(4)(x_i)$
 $n ? 1 j = 1 ? + a_4(x_i) ?w ?? (x_i) + ? ? j B_j ?(4)(x_i) n ? 1 j = 1 ? + a_5(x_i) ?w ? (x_i) + ? ? j B_j ?(4)(x_i)$
 $n ? 1 j = 1 ? + ?? 6 (?? ??) ???(?? ??) + ? ?? ?? ?? ? ?? (?? ??) ???1 ?? = 1 ? I 2012 March ? j$
 $~(?? P_j ?) P' j (? o) P -1 (?), for j = 0, 1, 2 P' -1 (? o) P_j(x), for j = 3, 4, ?, n-3 P_j(x) -P' j n) P_{n+1} for$
 $j = n-2, n-1, n. P'_{n+1} \{ (? n) (? ?), - Q_j (? - Q'' j (? o) o (? for j = 1, 2 Q'' o (? o) Q_j (?), for j = 3, 4, ? , n-3$
 $Q_j (? - Q'' j (? n n (?), for j = n-2, n-1, Q'' n (? n) Q),) Q \{ ? 2 (? ? 1 (? + A_1 - ? . 1 (? o) -1 (? + 1 - ? .$
 $1 (? n) P_{n+1} (?) P' -1 (? o) P_{n+1} (? n) = P B' = ??(?? ??) for i = 1, 2, ?, n-1.(16)$
 Rearranging the terms and writing the system of equations (16) in matrix form, we get $A ? = B (17)$ where A
 $= ? a_{ij} ?; a_{ij} = B_j ?(4)(x_{i?1}) ? a_0(x_i) h^2 ? a_1(x_i) 2h ? + B_j ?(4)(x_i) ??? a_0(x_i) h^2 ? + B_j ?(4)(x_{i+1})$
 $? a_0(x_i) h^2 + a_1(x_i) 2h ? + B_j ?(4)(x_i) a_2(x_i) + B_j ??? (x_i) a_3(x_i) + B_j ?(4)(x_i) a_4(x_i) + B_j ?(4)(x_i) a_5(x_i) + B_j ?(4)(x_i) a_6(x_i)$
 $for i = 1, 2, ? , n ? 1, j = 1, 2, ? , n ? 1.(18) B = [b_i];$
 $b_i = b(x_i) - [w(4)(x_{i?1}) ? a_0(x_i) h^2 ? a_1(x_i) 2h ? + w(4)(x_i) ??? a_0(x_i) h^2 ? + w(4)(x_{i+1}) ?$
 $a_0(x_i) h^2 + a_1(x_i) 2h ? + w(4)(x_i) a_2(x_i) + w ??? (x_i) a_3(x_i) + w ?? (x_i) a_4(x_i) + w ? (x_i) a_5(x_i) + w(x_i) a_6(x_i)]$
 $for i = 1, 2, ? , n ? 1.(19)$
 and $? = ?? 1, ? 2, ? , ? n ? 1, ? T .$

7 V. SOLUTION PROCEDURE TO FIND THE NODAL PARAMETERS

The basis function $B_i ?(x)$ is defined only in the interval $[x_{i?3}, x_{i+3}]$ and outside of this interval it is zero. Also at the end points of the interval $[x_{i?3}, x_{i+3}]$ the basis function $B_i ?(x)$ vanishes. Therefore, $B_i ?(x)$ is having non-vanishing values at the mesh points $x_{i?2}, x_{i?1}, x_i, x_{i+1}, x_{i+2}$ and zero at the other mesh points. The first four derivatives of $B_i ?(x)$ also have the same nature at the mesh points as in the case of $B_i ?(x)$.

Using these facts, we can say that the matrix A defined in (18) is a seven diagonal band matrix. Therefore, the system of equations (???) is a seven diagonal band system in $s_i ?$. The nodal parameters $s_i ?$ can be obtained by using band matrix solution package. We have used the FORTRAN-90 programming to solve the boundary value problem (1)-(2) by the proposed method.

8 VI.

9 NUMERICAL EXAMPLES

To demonstrate the applicability of the proposed method for solving the sixth order boundary value problems of type (1)-(2) we considered seven examples of which four are linear and three are non linear boundary value problems. Numerical results for each problem are presented in tabular forms and compared with the exact solutions or numerical solutions available in the literature.

Example 1 : Consider the linear boundary value problem $y(6) + xy = ?(24 + 11x + x^3)e^x, 0 < ?? < 1$ (20) subject to $y(0) = 0, y(1) = 0, y'(0) = 1, y'(1) = ?e, (21)$
 $y''(0) = 0, y''(1) = ?4e.$

The exact solution for the above problem is given by $y(x) = x(1 - x)e^x$. The proposed method is tested on this problem where the domain $[0,1]$ is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 1 subject to (23). The exact solution for the above problem is given by $y(x) = x^3(1 - x)^3$. The proposed method is tested on this problem where the domain $[0,1]$ is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 2. The maximum absolute error obtained by the proposed method is 2.942979×10^{-6} .

$y + y''' + y'' - y = e^x$ ($15x^2 + 78x - 114$), $0 < x < 1$
subject to $y(0) = 0$, $y(1) = 1$, $y'(0) = 0$, $y'(1) = 2e$, (25)

The exact solution for the above problem is given by $y(x) = x^3 e^x$. The proposed method is tested on this problem where the domain $[0,1]$ is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 3. The maximum absolute error obtained by the proposed method is 4.939735×10^{-6} . (6) $+ y = 6(2x \cos x + 5 \sin x)$, $0 < x < 1$ (26) subject to $y(0) = 0$, $y(1) = 0$, $y'(0) = 2 \sin 1$, $y'(1) = 2 \sin 1$, $y''(0) = 4 \cos 1 + 2 \sin 1$, (27)

$y''(1) = 4 \cos 1 + 2 \sin 1$.

The exact solution for the above problem is given by $y(x) = (x^2 - 1) \sin x$. The proposed method is tested on this problem where the domain $[-1,1]$ is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 4. The exact solution for the above problem is given by $y(x) = \sin x$. Here $y_{(n+1)}$ is the $(n+1)$ th approximation for y . The domain $[0,1]$ is divided into 10 equal subintervals and the proposed method is applied to the sequence of problems (34). Numerical results for this problem are presented in Table 6. The maximum absolute error obtained by the proposed method is 1.913309×10^{-5} . (0) $= 1$, $y(1) = e^{-1/2}$, $y'(0) = 1/2$, $y'(1) = 1/2 e^{-1/2}$, (37) $y''(0) = 1/4$, $y''(1) = 1/4 e^{-1/2}$.

This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique [20] as Here $y_{(n+1)}$ is the $(n+1)$ th approximation for y . The domain $[0,1]$ is divided into 10 equal subintervals and the proposed method is applied to the sequence of problems (38). The exact solution for the problem (36) and (37) is not available in the literature. The numerical solutions can be obtained for this problem by refining the mesh size. Hussin and Kilicman [25] obtained the numerical solutions for this problem by refining the mesh size. Numerical results obtained by the proposed method are compared with the numerical results obtained by Hussin and Kilicman [25], and the results are presented in Table 7. $y_{(n+1)}(6) + 3e^{xy(n)} y_{(n+1)} = 2e^{xy(n)} y_{(n+1)}^3$ for $n = 0, 1, 2$, (38)

10 CONCLUSION

In this paper, we have developed a collocation method with quintic B-splines as basis functions to solve sixth order boundary value problems. Here we have taken internal mesh points as the selected collocation points. The quintic B-spline basis set has been redefined into a new set of basis functions which in number match with the number of selected collocation points. The proposed method is applied to solve several number of linear and non-linear problems to test the efficiency of the method. The numerical results obtained by the proposed method are in good agreement with the exact solutions or numerical solutions available in the literature. The objective of this paper is to present a simple method to solve a sixth order boundary value problem and its easiness for implementation.

1. Glatzmaier, G. A. (1985). Numerical simulations of stellar convection dynamics at the base of the convection zone. *Geophys. Fluid Dynamics*, 31, 137-150.

11 Numerical

Absolute error by solutions obtained by Hussin and Kilicman [25] the proposed method when compared with Hussin and Kilicman [25] 0^{1 2}

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Figure 1: (6)



Figure 2: +?? 3 (



Figure 3:

$$\frac{A_x \cdot w_x(x_{\alpha})}{a_x^*(x_{\alpha})} \varrho_{\alpha}(x) + \frac{B_x \cdot w_x(x_{\alpha})}{a_x^*(x_{\alpha})} \varrho_{\alpha}(x)$$

Figure 4:

such that

Now the quintic B-splines ?? ?? (??) are defined by

where

1

[Note: Numerical results for Example 1 Example 2: Consider the linear boundary value problem]

2

3

4

6

4

	Exact solution	Absolute error by proposed method
-0.8	2.582482E-01	4.172325E-07
-0.6	3.613712E-01	1.251698E-06
-0.4	3.271114E-01	1.788139E-07
-0.2	1.907225E-01	2.488494E-06
0.0	0.000000	5.105962E-06
0.2	-1.907226E-01	8.076429E-06
0.4	-3.271114E-01	7.987022E-06
0.6	-3.613712E-01	6.377697E-06
0.8	-2.582482E-01	3.904104E-06

Figure 10: Table 4 :

6

	Exact solution	Absolute error by proposed method
0.1	9.048374E-01	1.907349E-06
0.2	8.187308E-01	7.033348E-06
0.3	7.408182E-01	1.347065E-05
0.4	6.703200E-01	1.877546E-05
0.5	6.065307E-01	1.913309E-05
0.6	5.488116E-01	1.478195E-05
0.7	4.965853E-01	8.702278E-06
0.8	4.493290E-01	3.188848E-06
0.9	4.065697E-01	2.980232E-08

Example 7 : Consider the nonlinear boundary value problem

$$y(6) = e \times y^3(x), \quad 0 < x < 1 \quad (36)$$

subject to

$$y$$

Figure 11: Table 6 :

7

Figure 12: Table 7 :

$40(1+x)^6$, for $n = 0, 1, 2$, ?

1 ??

(1) = ?1 24 .

Here $y_{(n+1)}$ is the $(n+1)$ th approximation for y . The domain $[0, 1]$ is divided into 10 equal subintervals and the proposed method is applied to the sequence of problems (30). Numerical results for this problem are presented in Table ?? . The maximum absolute error obtained by the proposed method is 1.329929×10^{-6} .

Table ?? : Numerical results for Example 5

Example 6 : Consider the nonlinear boundary value problem $y'' + e^x y^2 = e^x + e^{3x}$, $0 < x < 1$ (32) subject to

$y(0) = 1$, $y(1) = 1$ e .

The exact solution for above problem is given by $y(x) = e^x$. This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique [20] as

$+ 2e^x y^{(n)} y^{(n+1)} = e^x y^2^{(n)} + e^x + e^{3x}$, for $n = 0, 1, 2$, ? (34) subject to

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