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About the Presence of Irregular Precession Motions in a Symmetric Euler Gyroscope

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About the Presence of Irregular Precession Motions in a Symmetric Euler Gyroscope

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I. Annotation

It is generally accepted that the only type of motion present in a symmetric Euler gyroscope (SEG) is regular precession. This paper proves that regular precession is not the only type of motion present, but corresponds only to the well-known initial coordinated Euler angles. At any other initial angles, motions that differ from regular precession occur. In the article, the problem is solved analytically in two stages: first, angular velocities of the gyroscope are determined using differential dynamic equations, at the second stage, as a result of integration of differential matrix kinematic and differential matrix Poisson equations (both with periodic coefficients), final relations about the SEG motion with arbitrary initial Euler angles are derived. Periodic coefficients are the SEG angular velocities that are found as a solution to the dynamic equations. From the obtained general formulas, special formulas of regular precession for particular coordinated initial Euler angles that coincide with the well-known ones are derived. For other initial angles, formulas for irregular precession are obtained. In addition to the solutions for the Euler angles, solutions for the Euler-Krylov angles were found, which in some cases provide a more explicit geometric interpretation of motion. The analytical results are supported by mathematical modeling. In particular, certain conditions were found – the "strong impact" condition when irregular SEG precession for the Euler-Krylov angles occurs in the direction of the rotational pulse, and the sign of the angular velocity of the gyroscope proper rotation changes to the opposite. At the Euler angles, the motions of irregular precession during the "strong" and "weak" impact conditions are qualitatively identical. In relation to the case of regular precession under the "strong" impact conditions, the changes are significant: the angles of precession and nutation become oscillatory, and the angular velocity and the angle of proper rotation change their sign to the opposite.

a) Relevance

Modern gyroscopic technology has achieved the highest accuracy in measuring angular motion parameters of moving objects (MO) in the field of classical symmetric Euler gyroscopes with electrostatic suspension. In the US Gravity Probe experiment, the four axially symmetric Euler gyroscopes with electrostatic cryogenic suspension mounted on the astronomical Earth satellite had values of drift angular velocities of less than 10^{-11} angular deg/hr. This, together with the telescope readings, experimentally confirms the Einsteinian general theory of relativity (GTR) by detecting a gyro axis shift with the accuracy of 1% equal to 6.6 angular seconds per year, which is effectively predicted by the GTR [1, 2]. It is noted that classical symmetric Euler gyroscopes (SEG) with electrostatic suspensions have drift angular velocities values of 10^{-5} angular deg/hr in terrestrial conditions, which is a better accuracy level than that of fiber optic (FOG) and laser (LG) gyroscopes, i.e. gyros based on new physical measurement principles in which drift angular velocities values are in the range of 10^{-4}-10^{-3} angular deg/hr, respectively [2]. Considering the fact that rotary classical Euler gyroscopes with magnetic active and magnetic resonance suspensions are still being developed and manufactured, it can be stated that studies concerning angular motions of the rotor's axis of proper rotation, which characterize its errors, are relevant. In this aspect, for a symmetric Euler gyroscope designed for GTR validation [1, 4], the parameters of its regular precession are evaluated, i.e. its errors, including the Poinsot analysis. A fundamental presentation of the theory of symmetric Euler gyroscopes with the Poinsot and McCullagh analyses of motion is given in [5–6].

It should be recalled that elementary particles - electrons, protons, etc. are essentially Euler gyroscopes [3] (one can say that the entire Universe consists of corpuscular Euler gyroscopes), which also emphasizes the relevance of this study.

b) Formulation of the problem

The solution to the problem of inertial motion of a symmetric Euler gyroscope is well known and described in many works, in particular, in [1-2]. This motion is regular precession, characterized by a constant angle of nutation between the kinetic moment axis, superimposed with the inertial basis axis, and the axis of SEG proper rotation. At the same time, the angular velocities of precession and nutation are constant.

The indicated properties have found application in [4] in the process of preparation of an experiment to validate the general theory of relativity using a SEG and

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a telescope on an artificial Earth satellite when solving the problem of selection of relations between the primary moments of inertia that provide very low angular precession velocities. In the experiment [1], drift angular velocities values were less than 10^{-11} angular deg/hr, which validated the Einsteinian general theory of relativity with an error of less than 1%.

It should be noted that the solution to the problem of regular precession was possible with the following restrictions on the initial Euler angles [6, formulas (2.39), (2.41)]:

$$\psi_0 = const; \quad \varphi_0 = 0; \qquad \cos\theta_0 = const = \frac{Cr_0}{G},$$

where G is the kinetic moment; r_0 is the SEG proper rotation angular velocity component; C is the primary moment of SEG inertia around the same axis.

This paper sets the task of finding the solution to the problem of SEG motion for arbitrary initial angles not only along the precession angle ψ_0, but also along the initial angles of nutation and proper rotation. The Poisson differential kinematic equations are used for this purpose. To clarify the problem formulation, let us cite a statement on this subject from the work [6, p. 79]. The first step in solving the problem is to determine the angular velocities of the body. This is solved analytically regardless of the Euler angles. The second step consists of determining the Euler angles by integrating the kinematic equations due to the angular velocities found in the first step. This long and arduous process is eased by applying the kinetic moment theorem and the method of selection of a coordinate system, one of the axes of which coincides with the kinetic moment vector [5-6], etc. For this article, we chose the way of integration of the matrix differential equations in quaternions, as well as of Poisson equations by means of solving the Cauchy problem with arbitrary initial angles, which is not related to the special selection of a coordinate system, one of the axes of which is directed along the kinetic moment vector of the SEG.

II. On the Influence of Initial Conditions for Kinematic Equations on the Nature of Motions in a Symmetric Euler Gyroscope

In this section, we set the task to clarify the range of values of the initial Euler angles for the kinematic equations of the symmetric Euler gyroscope, with which they are reduced to identities - after substituting their analytical solutions given in [7], as well as the solutions of dynamic equations given in [6]. Since these solutions describe regular precession, we are talking about the initial conditions under which it is observed, and under which it is not.

Dynamic equations for a symmetric Euler gyroscope have the form [7, p. 126]:

$$\begin{aligned}
&A\frac{dp}{dt} + (C - A)qr = 0 \\
&A\frac{dq}{dt} + (A - C)rp = 0 \\
&C\frac{dr}{dt} = 0; \quad \frac{dr}{dt} = 0; \quad r = r_0 = const
\end{aligned} \tag{A.1}$$

The kinematic Euler equations [7, p. 115]:

$$\left.\begin{aligned}
p &= \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\
q &= \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \\
r &= \dot{\psi}\cos\theta + \dot{\varphi}
\end{aligned}\right\} \tag{A.2}$$

The solutions of these equations obtained in [7, p. 37]:

$$\left.\begin{aligned}
&\psi = nt + \psi_0; \qquad n = \frac{G}{A} \\
&\cos\theta = \frac{Cr}{G} = \frac{Cr_0}{G}; \quad \theta = \theta_0 = const \\
&\varphi = n_1 t + \varphi_0; \qquad n_1 = r_0 - n\cos\theta_0
\end{aligned}\right\} \tag{A.3}$$

Where r_0, G are the constants, $G^2 = \left(A^2 p^2 + A^2 q^2\right) + C^2 r_0^2$.

Equations (A.2), resolved in relation to $\dot{\psi}$, $\dot{\theta}$, $\dot{\varphi}$ [[6, p. 46]:

$$\left.\begin{aligned} \frac{d\psi}{dt} &= \left(p\sin\varphi + q\cos\varphi\right)\Big/\sin\theta \\ \frac{d\theta}{dt} &= \left(p\cos\varphi - q\sin\varphi\right) \\ \frac{d\varphi}{dt} &= r - ctg\theta\cdot\left(p\sin\varphi + q\cos\varphi\right) \end{aligned}\right\} \quad (A.4)$$

Considering the solutions of equations (A.1) and the designations from [6, p. 88] we have:

$$\begin{aligned} &p = \omega_1 = \omega_{10}\sin\nu t; \quad && q = \omega_2 = \omega_{10}\cos\nu t; \quad && r = r_0 = \omega_{30}; \\ &G^2 = A^2\omega_{10}^2 + C^2\omega_{30}^2; && G = H; && \dot{\varphi} = n_1 = \nu; \\ &n = \dot{\psi}; && \sin\theta_0 = \frac{A\omega_{10}}{G}; && \tan\theta_0 = \frac{A\omega_{10}}{C\omega_{30}}. \end{aligned} \quad (A.5)$$

Substitution of solutions into equations (A.4)

Substituting (A.5) into (A.4), we obtain, in consideration of (A.3) for the third equation in (A.4):

$$r_0 - \frac{G}{A}\cos\theta_0 = r_0 - \omega_{10}\frac{\sin\dot{\varphi}t\cdot\sin(\dot{\varphi}t+\varphi_0) + \cos\dot{\varphi}t\cdot\cos(\dot{\varphi}t+\varphi_0)}{\tan\theta_0}$$

$$-\frac{G}{A}\sin\theta_0 = -\omega_{10}\cos(\dot{\varphi}t + \varphi_0 - \dot{\varphi}t) = -\omega_{10}\cos\varphi_0 \quad (A.6)$$

$$-\frac{G}{A}\cdot\frac{A\omega_{10}}{G} = -\omega_{10}\cos\varphi_0 \approx -\omega_{10} = -\omega_{10}\cos\varphi_0$$

The equality (A.6) is reduced to an identity when

$$\varphi_0 = 0;\ \pm 2\pi m \ \left(m = 1,2,3,\ldots\right). \quad (A.7)$$

That is, the angle φ_0 should be zero. For the equation (A.2) of the system (A.4) we have:

$$0 = \omega_{10}\left(\sin\dot{\varphi}t\cdot\cos\left(\dot{\varphi}t+\varphi_0\right) - \cos\dot{\varphi}t\cdot\sin\left(\dot{\varphi}t+\varphi_0\right)\right)$$

$$0 = \omega_{10}\sin\left(\dot{\varphi}t - \dot{\varphi}t - \varphi_0\right) = -\omega_{10}\sin\left(\varphi_0\right) \quad (A.8)$$

The equality (A.8) is reduced to an identity at the angles $\varphi_0 = 0$; $\pm\pi m$ $\left(m = 1,2,3,\ldots\right)$.

For the angle $\psi_0 + nt$ from the first equation in (A.4) we have:

$$n = \left(p\sin\varphi + q\cos\varphi\right)\Big/\sin\theta = G\big/A \quad (A.9)$$

In consideration of (A.3) and (A.5) we obtain:

$$\frac{G}{A}\sin\theta_0 = \omega_{10}\cos\left(\dot{\varphi}t + \varphi_0 - \dot{\varphi}t\right) = \omega_{10}\cos\varphi_0$$

$$\frac{G}{A}\cdot\frac{A\omega_{10}}{G} = \omega_{10}\cos\varphi_0 \Rightarrow \omega_{10} = \omega_{10}\cos\varphi_0$$

That is, we obtain the relations (A.7) once again. This means that the equation (A.9) is reduced to an identity with $\theta = \theta_0$; $\varphi = n_1 t$ for any value of ψ_0. From (A.9) it also follows that when $\Delta\theta(0)$ of the initial value of the angle θ_0 is varied, i.e. for $\theta = \theta_0 + \Delta\theta(0)$, the equality (A.9) is not reduced to an identity.

From these calculations, we conclude that regular precession in a symmetric Euler gyroscope is possible only with the following values of the initial angles:

$$\psi_0 = const; \qquad \varphi_0 = 0; \qquad \cos\theta_0 = const = {Cr_0}/{G} \tag{A.10}$$

With any other initial values of the Euler angles, the equations (A.4) are reduced to identities with other solutions that do not coincide with the functions (A.3).

The relevance of the article is further reinforced by publications [4-7].

III. Problem Solution

a) Quaternion problem solution

Instead of integrating the degenerate Euler equations

$$\left.\begin{aligned} \frac{d\psi}{dt} &= {(p\sin\varphi + q\cos\varphi)}/{\sin\theta} \\ \frac{d\theta}{dt} &= (p\cos\varphi - q\sin\varphi) \\ \frac{d\varphi}{dt} &= r - ctg\theta\cdot(p\sin\varphi + q\cos\varphi) \end{aligned}\right\}, \tag{1}$$

In this article, we use the method of integration of quaternion and Poisson matrices that are non-degenerate for any angle value of the equation:

$$2\frac{dN^1}{dt} = P(t)N^1; \qquad N^1(0) = E. \ \frac{dA^1}{dt} = P(t)A^1; \quad A^1(t) = E; \tag{2}$$

The choice of the two types of equations is related to their widespread use in science and technology, it also enables comparison of their solutions. The coefficients and variables included in the differential equations (2) are indicated below.

Following [6], we present the Euler rotation angles diagram depicting the inertialess frames of the cardan suspension according to Fig. 1. Let us associate the moving coordinate system Oxyz (corresponds to the coordinate system O1'2'3' in [6]) with the gyroscope body, and also introduce inertial coordinate systems: the expanded Oξηζ, system, which coincides with the coordinate system Oxyz at the initial moment, and the original system $O\xi_H\eta_H\zeta_H$, relative to which the coordinate system Oξηζ is rotated at the initial angles Ψ_o, Θ_o, Φ_o. Figure 2 shows a similarly constructed diagram of the same gyroscope, but for the Euler - Krylov angles (ψ, Θ, φ).

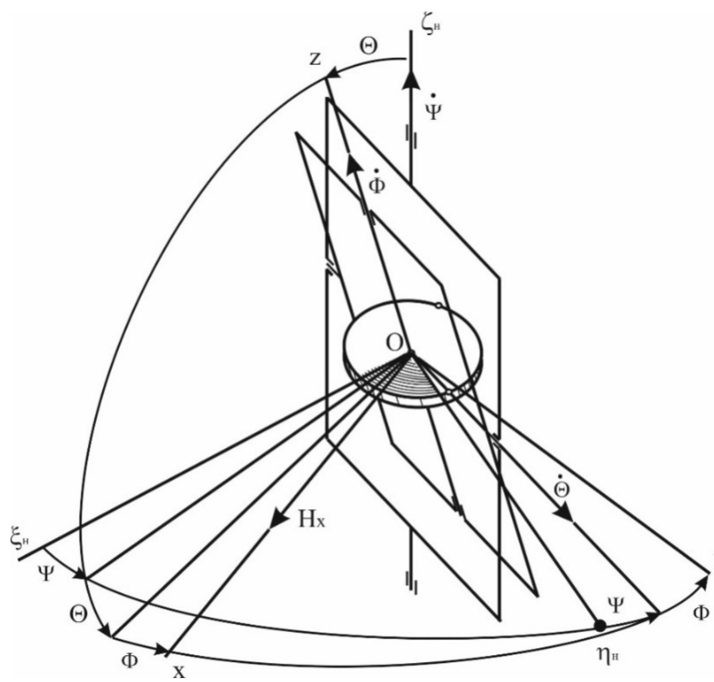


Fig.1

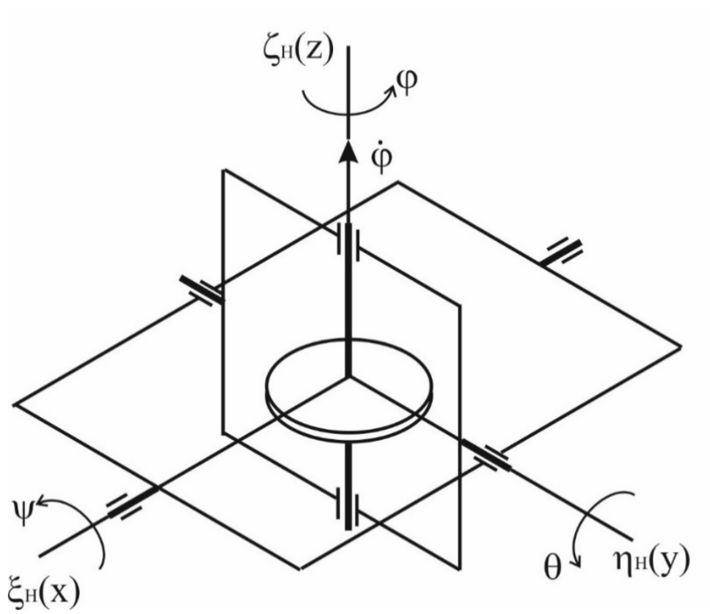


Fig. 2

The following scheme [12] corresponds to the rotation diagram of the introduced systems according to Fig. 1:

$$\xi_H\eta_H\zeta_H \xrightarrow[\Psi_0\Theta_0\Phi_0]{N_0,A_0} \xi\eta\zeta \xrightarrow[\Psi_1\Theta_1\Phi_1]{N^1,A^1} xyz \sim \xi_H\eta_H\zeta_H \xrightarrow[\Psi\Theta\Phi]{N,A} xyz$$

$$N = N^1 N_0, \tag{3}$$

where Ψ_o, Θ_o, Φ_o, N_o=N(0) are the initial angles of SEGrotation and the corresponding quaternion matrix [10, 11]; Ψ_1, Θ_1, Φ_1, N^1 are the rotation angles corresponding to the matriciant N^1, when N_o=E (E is the identity matrix); Ψ, Θ, Φ, N are the angles of the resulting rotation and the corresponding quaternion matrix of the resulting rotation.

Following the technique described in [8, 9] for matrices of directional cosines, we find the analytical solution for the quaternion matrix N^1 based on kinematic equations. Note that the quaternion matrices are related to the matrices of directional cosines of the angles by the relation $A = M^T N$ [10, 11]. In the article [8], the formulas for the angular velocities p, q, r of the gyroscope are solutions of the dynamic equations of the SEG, which had the initial angular velocity p(0)=q(0)=0; r(0)=R, and which was affected by the impact to the axis of the gyroscope figure in the form of a rotational pulse M_o around the axis Ox (hereinafter, $M_o = H_x$ is the kinetic moment from the impact). The dynamic Euler equations for a gyroscope with a dynamic axis of symmetry have the following form [8]:

$$\left.\begin{aligned} &\frac{dp}{dt} + \Omega q = \frac{M_0}{A}\cdot\frac{d}{dt}[I(t)] \\ &\frac{dq}{dt} - \Omega p = 0 \\ &\frac{dr}{dt} = 0;\ \ \Omega = r\cdot\frac{C-A}{A} \end{aligned}\right\} \tag{4}$$

p, q, r are the components of the vector of angular velocity of rotation of the gyroscope in the axes associated with it; I (t) is the unit function.
For initial conditions

t=0; p(0)=0; q(0)=0; r(0)=R

The solution to the system of differential equations (2) has the following form:

$$p = a\cos\Omega t;\quad q = a\sin\Omega t;\quad r = R;\quad a = \frac{M_0}{A} = \frac{H_x}{A}. \tag{5}$$

The transformation of coordinate systems from theinertial Oξηζ to the moving Oxyz, in consideration of the initial inertial coordinate system $O\xi_н\eta_н\zeta_н$, according to (3) is determined by the relations:

$$[xyz]^T = A^1 A(0)[\xi_н\eta_н\zeta_н]^T = A[\xi_н\eta_н\zeta_н]^T$$

or, equivalently, through quaternion matrices [10], [11]:

$$[xyz]^T = N^1 M^{1T} M^T(0) N(0)[\xi_н\eta_н\zeta_н]^T; \tag{6}$$

$$N^1 = N^\Phi N^\Theta N^\Psi;\quad N = N^1 N(0);\quad A^1 = M^{1T} N^1;\quad A(0) = M^T(0)N(0),$$

where N, A are the quaternion matrix and the matrix of directional cosines of the resulting rotation; N^1, A^1 are the matriciants; N^Φ, N^Θ, N^Ψ are the quaternion matrices of the corresponding simplest rotations. At the same time, M and N are the corresponding types of quaternion matrices [10, 11].

The matrix of directional cosines of the Euler angles for Fig. 1 when combining the coordinate systems $\xi\eta\zeta$ and $\xi_H\eta_H\zeta_H$:

$$A^1 = \begin{bmatrix} \cos\Psi_1\cos\Theta_1\cos\Phi_1 - \sin\Psi_1\sin\Phi_1 & \sin\Psi_1\cos\Theta_1\cos\Phi + \cos\Psi_1\sin\Phi_1 & -\sin\Theta_1\cos\Phi_1 \\ -\cos\Psi_1\cos\Theta_1\sin\Phi_1 - \sin\Psi_1\cos\Phi_1 & -\sin\Psi_1\cos\Theta_1\sin\Phi_1 + \cos\Psi_1\cos\Phi_1 & \sin\Theta_1\sin\Phi_1 \\ \cos\Psi_1\sin\Theta_1 & \sin\Psi_1\sin\Theta_1 & \cos\Theta_1 \end{bmatrix}. \tag{7}$$

The matrix of directional cosines of the Euler-Krylov angles (Fig. 2), which is equal to the matrix (7), has the form:

$$A^k = \begin{bmatrix} \cos\varphi\cos\theta & \sin\psi\sin\theta\cos\varphi + \cos\psi\sin\varphi & -\cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi \\ -\sin\varphi\cos\theta & -\sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi & \cos\psi\sin\theta\sin\varphi + \sin\psi\cos\varphi \\ \sin\theta & -\sin\psi\cos\theta & \cos\theta\cos\psi \end{bmatrix}. \tag{8}$$

The matrix N^1 corresponding to $N^1(0)=E$, i.e. to theangles Ψ(0)=Θ(0)=Φ(0)=0 (that is, the matriciant), can be determined by integrating the quaternion matrix equation [10, 11]:

$$2\frac{dN^1}{dt} = P(t)N^1;\qquad N^1(0) = E. \tag{9}$$

$$P(t)=\begin{bmatrix} 0 & -p & 0 & -r \\ p & 0 & r & 0 \\ 0 & -r & 0 & p \\ r & 0 & -p & 0 \end{bmatrix}; \quad N^1=\begin{bmatrix} \lambda_0^1 & -\lambda_1^1 & -\lambda_2^1 & -\lambda_3^1 \\ \lambda_1^1 & \lambda_0^1 & \lambda_3^1 & -\lambda_2^1 \\ \lambda_2^1 & -\lambda_3^1 & \lambda_0^1 & \lambda_1^1 \\ \lambda_3^1 & \lambda_2^1 & -\lambda_1^1 & \lambda_0^1 \end{bmatrix}.$$

The angular velocity matrix in consideration of (5) has the form:

$$P(t)=\begin{bmatrix} 0 & -a\cos\Omega t & -a\sin\Omega t & -R \\ a\cos\Omega t & 0 & R & -a\sin\Omega t \\ a\sin\Omega t & -R & 0 & a\cos\Omega t \\ R & a\sin\Omega t & -a\cos\Omega t & 0 \end{bmatrix}, \tag{10}$$

Which means that it satisfies the condition P(t)=P(t+τ); τ=2π/Ω.
Therefore, the system (9) is Lyapunov reducible [8]. By means of substitution

$$N_Z=N_\Phi N^1 \tag{11}$$

The system (9), (10) is reduced to an equivalent differential equation with constant coefficients

$$\frac{dN_Z}{dt}=P_B N_Z. \tag{12}$$

$$N_\Phi=\begin{bmatrix} \nu_0 & -\nu_1 & -\nu_2 & -\nu_3 \\ \nu_1 & \nu_0 & \nu_3 & -\nu_2 \\ \nu_2 & -\nu_3 & \nu_0 & \nu_1 \\ \nu_3 & \nu_2 & -\nu_1 & \nu_0 \end{bmatrix}; \quad P_B=\begin{bmatrix} 0 & -a & 0 & -R_1 \\ a & 0 & R_1 & 0 \\ 0 & -R_1 & 0 & a \\ R_1 & 0 & -a & 0 \end{bmatrix}. \tag{13}$$

$$R_1=R\frac{C}{A}; \quad \nu_0=\cos{\Omega t}/{2}; \quad \nu_1=\nu_2=0; \quad \nu_3=\sin{\Omega t}/{2}.$$

Given these formulas, we have:

$$N_\Phi=\begin{bmatrix} \cos{\Omega t}/{2} & 0 & 0 & -\sin{\Omega t}/{2} \\ 0 & \cos{\Omega t}/{2} & \sin{\Omega t}/{2} & 0 \\ 0 & -\sin{\Omega t}/{2} & \cos{\Omega t}/{2} & 0 \\ \sin{\Omega t}/{2} & 0 & 0 & \cos{\Omega t}/{2} \end{bmatrix}. \tag{14}$$

The equivalence of equations (9) and (12), (13) is confirmed by the fulfillment of the identity

$$N_\Phi\left(PN_\Phi^{-1}-\dot{N}_\Phi^{-1}\right)\equiv P_B \tag{15}$$

The solution to the equation (12) with constant coefficients is the Cauchy formula:

$$N_Z=L(t)L^{-1}(0)N_Z(0), \tag{16}$$

where L(t) is the fundamental matrix of solutions; $N_Z(0)$ is the matrix of initial values of the angles, equal, by condition, to the identity matrix: $N_Z(0)=E$.

After finding the fundamental matrix of solutions and a number of transformations, let us write down the expression (16) in the form:

$$N_Z = \left(E\cos\frac{\chi}{2} + D\frac{\sin \chi/2}{\chi} \right) N_Z(0);$$

$$D = \int_0^t P_B(\tau)d\tau; \chi = \left(\chi_1^2 + \chi_3^2\right)^{1/2} = \left(a^2 + R_1^2\right)^{1/2} t = n\,;t \tag{17}$$

$$\chi_1 = \int_0^t a(\tau)d\tau; \quad \chi_3 = \int_0^t R_1(\tau)d\tau; \quad \left(a^2 + R_1^2\right)^{1/2} = n; \quad \chi = nt.$$

After transformations, the matriciant takes the form:

$$N_Z = \begin{bmatrix} \cos \chi/2 & -\frac{a}{n}\sin \chi/2 & 0 & -\frac{R_1}{n}\sin \chi/2 \\ \frac{a}{n}\sin \chi/2 & \cos \chi/2 & \frac{R_1}{n}\sin \chi/2 & 0 \\ 0 & -\frac{R_1}{n}\sin \chi/2 & \cos \chi/2 & \frac{a}{n}\sin \chi/2 \\ \frac{R_1}{n}\sin \chi/2 & 0 & -\frac{a}{n}\sin \chi/2 & \cos \chi/2 \end{bmatrix} \tag{18}$$

From the expression (11) we have:

$$N^1 = N_\Phi^T N_Z; N = N_\Phi^T N_Z N(0) = N^1 N(0); \quad N^1 = N^1(\lambda_{a\kappa}) \text{ (k=0,1,2,3).} \tag{19}$$

In consideration of (13), (14) and (18), the expanded expression for the quaternion matriciant N[1] is derived below.

Since $N = N^1 N(0) = N_\Phi^T N_Z N(0)$, we have the following expression for the quaternion matrix of the resulting rotation N for nonzero initial conditions:

$$N = \begin{bmatrix} n_{a0} & -n_{a1} & -n_{a2} & -n_{a3} \\ n_{a1} & n_{a0} & n_{a3} & -n_{a2} \\ n_{a2} & -n_{a3} & n_{a0} & n_{a1} \\ n_{a3} & n_{a2} & -n_{a1} & n_{a0} \end{bmatrix} \cdot \begin{bmatrix} n_{00} \\ n_{01} \\ n_{02} \\ n_{03} \end{bmatrix}.$$

Formulas for the components of the quaternion matrix N:

$$\left.\begin{aligned} n_0 &= n_{a0}\cdot n_{00} - n_{a1}\cdot n_{01} - n_{a2}\cdot n_{02} - n_{a3}\cdot n_{03} \\ n_1 &= n_{a1}\cdot n_{00} + n_{a0}\cdot n_{01} + n_{a3}\cdot n_{02} - n_{a2}\cdot n_{03} \\ n_2 &= n_{a2}\cdot n_{00} - n_{a3}\cdot n_{01} + n_{a0}\cdot n_{02} + n_{a1}\cdot n_{03} \\ n_3 &= n_{a3}\cdot n_{00} + n_{a2}\cdot n_{01} - n_{a1}\cdot n_{02} + n_{a0}\cdot n_{03} \end{aligned}\right\} \tag{20}$$

By marking

$$\lambda_{a_i} = n_{ai} \quad \left(i = \overline{0,3}\right), \tag{21}$$

We have the explicit form of the formulas for the components of the quaternion matriciant N[1]:

$$\begin{aligned}
n_{a0} &= \lambda_{a0} = \cos\frac{\Omega t}{2}\cos\frac{nt}{2} + \frac{R_1}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2} \\
n_{a1} &= \lambda_{a1} = \frac{a}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2} \\
n_{a2} &= \lambda_{a2} = \frac{a}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2} \\
n_{a3} &= \lambda_{a3} = -\sin\frac{\Omega t}{2}\cos\frac{nt}{2} + \frac{R_1}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}
\end{aligned} \quad , \tag{22}$$

For regular precession, the angles of the initial orientation and the components of the initial quaternion are expressed by the formulas:

$$\begin{aligned}
&\Psi_0 = 0; \quad \Phi_0 = 0; \quad \Theta(0) = \Theta_0; \quad tg\Theta_0 = -{H_x}/{H}; \\
&\lambda_0 = n_{00} = \cos\frac{\Theta_0}{2}; \quad \lambda_1 = n_{01} = 0; \quad \lambda_2 = n_{02} = \sin{\Theta_0}/{2}; \quad \lambda_3 = n_{03} = 0.
\end{aligned} \tag{23}$$

In this regard, we have:

$$\begin{aligned}
n_0 &= n_{a0}\cos\frac{\Theta_0}{2} - n_{a2}\sin\frac{\Theta_0}{2} \\
n_1 &= n_{a1}\cos\frac{\Theta_0}{2} + n_{a3}\sin\frac{\Theta_0}{2} \\
n_2 &= n_{a2}\cos\frac{\Theta_0}{2} + n_{a0}\sin\frac{\Theta_0}{2} \\
n_3 &= n_{a3}\cos\frac{\Theta_0}{2} - n_{a1}\sin\frac{\Theta_0}{2}
\end{aligned}$$

In consideration of (23) we obtain:

$$\begin{aligned}
n_0 &= \cos\frac{\Omega t}{2}\cos\frac{nt}{2}\cos\frac{\Theta_0}{2} + \frac{R_1}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\cos\frac{\Theta_0}{2} - \frac{a}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\sin\frac{\Theta_0}{2} \\
n_1 &= \frac{a}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\cos\frac{\Theta_0}{2} - \sin\frac{\Omega t}{2}\cos\frac{nt}{2}\sin\frac{\Theta_0}{2} + \frac{R_1}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\sin\frac{\Theta_0}{2} \\
n_2 &= \frac{a}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\cos\frac{\Theta_0}{2} - \sin\frac{\Omega t}{2}\cos\frac{nt}{2}\sin\frac{\Theta_0}{2} + \frac{R_1}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\sin\frac{\Theta_0}{2} \\
n_3 &= -\sin\frac{\Omega t}{2}\cos\frac{nt}{2}\cos\frac{\Theta_0}{2} + \frac{R_1}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\cos\frac{\Theta_0}{2} - \frac{a}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\sin\frac{\Theta_0}{2}
\end{aligned} \tag{24}$$

After that, let us similarly determine the trigonometric functions for the Euler-Krylov angles ψ, Θ, φ on the basis of the matrix (8) and its quaternion counterpart [8, 9]. We have:

$$tan\psi = -\frac{a_{32}}{a_{33}} = \frac{2(\lambda_0\lambda_1 - \lambda_2\lambda_3)}{\lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2}$$

$$\sin\theta = a_{31} = 2(\lambda_0\lambda_2 + \lambda_1\lambda_3)$$

$$\tan\varphi = -\frac{a_{21}}{a_{11}} = \frac{2(\lambda_0\lambda_3 - \lambda_1\lambda_2)}{\lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2} \tag{25}$$

Substituting the quaternion components (22) into these formulas, we obtain

$$\tan\psi = \frac{\frac{a}{n}\sin n\,t}{\frac{a^2}{n^2}\cos n\,t + {R_1^2}/{n^2}};$$

$$\sin\theta = \frac{aR_1}{n^2}(1-\cos n\,t);$$

$$\tan\varphi = \frac{-\sin\Omega t\left(\frac{R_1^2}{n^2}\cos nt + \frac{a^2}{n^2}\right) + \frac{R_1}{n}\sin nt\cos\Omega t}{\cos\Omega t\left(\frac{R_1^2}{n^2}\cos nt + \frac{a^2}{n^2}\right) + \frac{R_1}{n}\sin nt\sin\Omega t} \tag{26}$$

These expressions coincide with formulas (18) [8], confirming the fidelity of the solutions to the problem for zero initial Euler-Krylov angles both in the quaternion form and in the form associated with the application of the Poisson differential kinematic equations.

For arbitrary initial Euler-Krylov angles, explicit solutions can be obtained from relations (24), (25) (in (25), the λ_i must be replaced by values n_i $\left(i=\overline{0,3}\right)$).

In turn, for the Euler angles we have the following solutions:

$$\tan\Psi = \frac{a_{32}}{a_{31}} = \frac{\lambda_2\lambda_3 - \lambda_0\lambda_1}{\lambda_0\lambda_2 + \lambda_1\lambda_3}$$

$$\cos\Theta = a_{33} = \lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2 \tag{27}$$

$$\tan\Phi = -\frac{a_{23}}{a_{13}} = \frac{\lambda_0\lambda_1 + \lambda_2\lambda_3}{\lambda_1\lambda_3 + \lambda_0\lambda_2}$$

In consideration of (22), we obtain the solutions in explicit form:

$$\tan\Psi = \frac{\frac{a}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\left(-\sin\frac{\Omega t}{2}\cos\frac{nt}{2} + \frac{R_1}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\right) - \frac{a}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\left(\cos\frac{\Omega t}{2}\cos\frac{nt}{2} + \frac{R_1}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\right)}{\frac{a}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\left(\cos\frac{\Omega t}{2}\cos\frac{nt}{2} + \frac{R_1}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\right) + \frac{a}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\left(-\sin\frac{\Omega t}{2}\cos\frac{nt}{2} + \frac{R_1}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\right)}$$

$$\begin{aligned}\cos\Theta = \Big(&\cos^2\frac{\Omega t}{2}\cos^2\frac{nt}{2} + 2\frac{R_1}{n}\sin\frac{\Omega t}{2}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\cos\frac{nt}{2} + \frac{R_1^2}{n^2}\sin^2\frac{\Omega t}{2}\sin^2\frac{nt}{2} + \sin^2\frac{\Omega t}{2}\cos^2\frac{nt}{2} - \\ &-2\frac{R_1}{n}\sin\frac{\Omega t}{2}\cos\frac{\Omega t}{2} + \frac{R_1^2}{n^2}\cos^2\frac{\Omega t}{2}\sin^2\frac{nt}{2} - \frac{a^2}{n^2}\cos^2\frac{\Omega t}{2}\sin^2\frac{nt}{2} - \frac{a^2}{n^2}\sin^2\frac{\Omega t}{2}\sin^2\frac{nt}{2}\Big)\end{aligned}$$

$$\tan\Phi = \frac{\frac{a}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\left(\cos\frac{\Omega t}{2}\cos\frac{nt}{2} + \frac{R_1}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\right) + \frac{a}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\left(-\sin\frac{\Omega t}{2}\cos\frac{nt}{2} + \frac{R_1}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\right)}{\frac{a}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\left(-\sin\frac{\Omega t}{2}\cos\frac{nt}{2} + \frac{R_1}{n}\cos\frac{\Omega t}{2}\sin\frac{nt}{2}\right) - \frac{a}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\left(\cos\frac{\Omega t}{2}\cos\frac{nt}{2} + \frac{R_1}{n}\sin\frac{\Omega t}{2}\sin\frac{nt}{2}\right)} \tag{28}$$

For regular precession in (24), (25), it is necessary to consider λ_i $\left(i=\overline{0,3}\right)$ according to the expressions (23), and then, after transformations, we obtain:

$$\Psi = \Psi_0 + nt;$$

$$\tan\Theta = \frac{aA}{\sqrt{(aA)^2 + CR_1^2}} = {a}/{n};$$

$$\Phi = \left(1 - {}^{C}/_{A}\right)Rt. \tag{29}$$

The result coincided with the classical one, which is expressed by the formulas (A.3, A.5).

Let us now consider a variant of the solution to the problem for irregular precession. It corresponds to the initial angles $\Phi_0 = \Psi_0 = 0; tan\Theta_0 = \frac{Aa}{CR}$ that differ from the angles (23), which generate regular precession, only by the sign of the angle of nutation. After transformations, the formulas for determining the Euler angles for the SEG are:

$$tan\Psi^* = -\frac{\sin n\,t}{2\cos^2\Theta_0 - \cos 2\,\Theta_0 \cos n\,t}$$

$$\cos\Theta^* = \frac{\cos\Theta_0 \cos 2\,\Theta_0}{tan^2\Theta_0} + 2\sin^2\Theta_0 \cos\Theta_0 \cos n\,t$$

$$tan\Phi^* = \frac{\sin\Theta_0 \cos 2\Theta_0 \sin\Omega t + \sin 2\Theta_0 \sin nt \cos\Omega t}{\sin\Theta_0 \cos 2\Theta_0 \cos\Omega t - 2\sin 2\Theta_0 \cos nt \cos\Omega t} \tag{30}$$

The expressions (30) suggest that only the change of the sign of the initial angle of nutation – with the other two initial angles unchanged – caused the appearance of irregular precession motions in the Euler gyroscope.

b) Solution for the Poisson matrix differential equation

The transformation of the coordinate system Oxyz from the initial position Oξηζ is characterized by the formulas:

$$[xyz]^T = A^1[\xi\eta\zeta]; \quad A^1 = A^\Phi A^\Theta A^\Psi, \tag{31}$$

Where A^Φ, A^Θ, A^Ψ are the transformation matrices of the coordinates of the simplest rotations. On the other hand, this matrix can be determined by integrating the Poisson matrix kinematic equation:

$$\frac{dA^1}{dt} = P(t)A^1; \quad A^1(t) = E; \tag{32}$$

$$A^1 = \begin{bmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 \\ a_{21}^1 & a_{22}^1 & a_{23}^1 \\ a_{31}^1 & a_{32}^1 & a_{33}^1 \end{bmatrix}; \quad P(t) = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} \tag{33}$$

The matrix of directional cosines of the Euler angles for Fig. 1 when combining the coordinate systems $\xi\eta\zeta$ and $\xi_H\eta_H\zeta_H$ – form (32), and the matrix of directional cosines of the Euler-Krylov angles (Fig. 2) – form (33). The angular velocity tensor for gyroscopes with a dynamic axis of symmetry has the form:

$$P(t) = \begin{Vmatrix} 0 & R & -a\sin\Omega t \\ -R & 0 & a\cos\Omega t \\ a\sin\Omega t & -a\cos\Omega t & 0 \end{Vmatrix}, \tag{34}$$

That is, it satisfies the condition $P(t) = P(t+\tau); \quad \tau = \dfrac{2\pi}{\Omega}$.

As a result of this condition, the system (32) - (33) is Lyapunov reducible [13]. Indeed, by substitution

$$Z = \Phi(t)A^1 \tag{35}$$

it is reduced to a matrix equivalent differential equation with constant coefficients

$$\frac{dZ}{dt} = BZ, \tag{36}$$

$$\Phi(t)=\begin{Vmatrix}\cos\Omega t & \sin\Omega t & 0\\ -\sin\Omega t & \cos\Omega t & 0\\ 0 & 0 & 1\end{Vmatrix};\quad B=\begin{Vmatrix}0 & R_1 & 0\\ -R_1 & 0 & a\\ 0 & -a & 0\end{Vmatrix};\quad Z=\left\|Z_{ij}\right\|;\quad R_1=R+\Omega;\quad (i,j)=1;2;3. \quad (37)$$

The equivalence of the equations (32) and (36) is confirmed by the validity of the identity $\Phi(t)\cdot\left(P\Phi^{-1}(t)-\Phi^{-1}(t)\right)\equiv B$. The differential linear homogeneous equation (36) is solved by the Cauchy formula

$$Z(t)=Q(t)\cdot Q^{-1}(0)\cdot Z(0), \quad (38)$$

Where Q(t) is the fundamental matrix of solutions; Z(0) is the matrix of initial values of directional cosines, and as provided by the condition, Z(0)=E. After finding the fundamental matrix and performing a number of transformations, the solution (38) takes the form:

$$Z=\begin{Vmatrix}\frac{R_1^2}{n^2}\cos nt+\frac{a^2}{n^2} & \frac{R_1}{n}\sin nt & -\frac{aR_1}{n^2}(1-\cos nt)\\ -\frac{R_1}{n}\sin nt & \cos nt & \frac{a}{n}\sin nt\\ \frac{aR_1}{n^2}(1-\cos nt) & -\frac{a}{n}\sin nt & \frac{a^2}{n^2}\cos nt+\frac{R_1^2}{n^2}\end{Vmatrix}; \quad (39)$$

$$n^2=a^2+R_1^2;\quad R_1=R\frac{C}{A}.$$

From (37) it follows that $A^1=\Phi^{-1}(t)\cdot Z$, as a result, the solution to the equation (32) for a gyroscope with a dynamic axis of symmetry is the matrix (matriciant):

$$A^1=\begin{Vmatrix}\cos\Omega t\left(\frac{R_1^2}{n^2}\cos nt+\frac{a^2}{n^2}\right)+ & \frac{R_1}{n}\cos\Omega t\cdot\sin nt- & \frac{aR_1}{n^2}(1-\cos nt)\cos\Omega t-\\ +\frac{R_1}{n}\sin nt\cdot\sin\Omega t & -\sin\Omega t\cdot\cos nt & -\frac{a}{n}\sin nt\sin\Omega t\\ \sin\Omega t\left(\frac{R_1^2}{n^2}\cos nt+\frac{a^2}{n^2}\right)- & \frac{R_1}{n}\sin\Omega t\cdot\sin nt+ & \frac{aR_1}{n^2}(1-\cos nt)\sin\Omega t+\\ -\frac{R_1}{n}\sin nt\cdot\cos\Omega t & +\cos\Omega t\cdot\cos nt & +\frac{a}{n}\sin nt\cos\Omega t\\ \frac{aR_1}{n^2}(1-\cos nt) & -\frac{a}{n}\sin nt & \frac{a^2}{n^2}\cos nt+\frac{R_1^2}{n^2}\end{Vmatrix}, \quad (40)$$

For the initial Euler angles, the matrix A_0 has the form:

$$A_0=\begin{bmatrix}\cos\Psi_0\cos\Theta_0\cos\Phi_0-\sin\Psi_0\sin\Phi_0 & \sin\Psi_0\cos\Theta_0\cos\Phi_0+\cos\Psi_0\sin\Phi_0 & -\sin\Theta_0\cos\Phi_0\\ -\cos\Psi_0\cos\Theta_0\sin\Phi_0-\sin\Psi_0\cos\Phi_0 & -\sin\Psi_0\cos\Theta_0\sin\Phi_0+\cos\Psi_0\cos\Phi_0 & \sin\Theta_0\sin\Phi_0\\ \cos\Psi_0\sin\Theta_0 & \sin\Psi_0\sin\Theta_0 & \cos\Theta_0\end{bmatrix}. \quad (41)$$

Formulas for determining the Euler angles:

$$\tan\Psi = \frac{a_{32}}{a_{31}} = \frac{\sum_{k=1}^{3} a_{3k}^{1} a_{k2}^{0}}{\sum_{k=1}^{3} a_{3k}^{1} a_{k1}^{0}}; \cos\Theta = a_{33} = \sum_{k=1}^{3} a_{3k}^{1} a_{k3}^{0}; \tan\Phi = -\frac{a_{23}}{a_{11}} = -\frac{\sum_{k=1}^{3} a_{2k}^{1} a_{k3}^{0}}{\sum_{k=1}^{3} a_{1k}^{1} a_{k1}^{0}}. \tag{42}$$

The following kinematic Euler equations correspond to the Poisson equations:

$$\left.\begin{array}{ll} \dot{\Psi} = \left(q\sin\Phi - p\cos\Phi\right)\big/\sin\Theta; & p = a\cos\Omega t; \\ \dot{\Theta} = p\sin\Phi + q\cos\Phi; & q = a\sin\Omega t; \\ \dot{\Phi} = r - \left(q\sin\Phi - p\cos\Phi\right)ctg\Theta; & r = R \end{array}\right\}. \tag{43}$$

$$t = t_0; \quad \Psi(t_0) = \Psi_0; \quad \Theta(t_0) = \Theta_0; \quad \Phi(t_0) = \Phi_0$$

Let us now apply the obtained formulas to the case of regular precession.
We use the initial values $\Phi_0 = \Psi_0 = 0; tan\Theta_0 = -\frac{Aa}{CR}$ in the matrix A_o associated with this type of precession

$$A_0 = \begin{bmatrix} \cos\Theta_0 & 0 & -\sin\Theta_0 \\ 0 & 1 & 0 \\ \sin\Theta_0 & 0 & \cos\Theta_0 \end{bmatrix}. \tag{44}$$

In consideration of this we obtain:

$$A = \begin{bmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 \\ a_{21}^1 & a_{22}^1 & a_{23}^1 \\ a_{31}^1 & a_{32}^1 & a_{33}^1 \end{bmatrix} \cdot \begin{bmatrix} \cos\Theta_0 & 0 & -\sin\Theta_0 \\ 0 & 1 & 0 \\ \sin\Theta_0 & 0 & \cos\Theta_0 \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11}^1\cos\Theta_0 + a_{13}^1\sin\Theta_0 & a_{12}^1 & -a_{11}^1\sin\Theta_0 + a_{13}^1\cos\Theta_0 \\ a_{21}^1\cos\Theta_0 + a_{23}^1\sin\Theta_0 & a_{22}^1 & -a_{21}^1\sin\Theta_0 + a_{23}^1\cos\Theta_0 \\ a_{31}^1\cos\Theta_0 + a_{33}^1\sin\Theta_0 & a_{32}^1 & -a_{31}^1\sin\Theta_0 + a_{33}^1\cos\Theta_0 \end{bmatrix} \tag{45}$$

$$\tan\Psi = \frac{a_{32}}{a_{31}} = \frac{a_{32}^1}{a_{31}^1\, cos\ \theta_0 + a_{33}^1\, sin\ \theta_0} \tag{46}$$

After conversion we obtain:

$$\tan\Psi = \frac{-\frac{a}{n} sin\ nt}{cos\ \theta_0 \cdot \frac{aR_1}{n^2}(1 - cos\ nt) + sin\ \theta_0 \cdot \left(\frac{a^2}{n^2} cos\ nt + \frac{R_1^2}{n^2}\right)} \tag{47}$$

$$\cos\Theta_0 = \frac{CR}{H}; \quad \sin\Theta_0 = -\frac{Aa}{H}. \tag{48}$$

$$\tan\Psi = \mathrm{tannt}; \Psi = \mathrm{nt} = \frac{\mathrm{H}}{\mathrm{A}}\mathrm{t}; \dot{\Psi} = \frac{\mathrm{H}}{\mathrm{A}} = \mathrm{n} \tag{49}$$

The solution (49) coincided with the classical one.
Let us now determine the value of the angle of nutation Θ:

$$cos\ \Theta = a_{33} = -a_{31}^1\, sin\ \Theta_0 + a_{33}^1\, cos\ \Theta_0$$

After calculations we obtain:

$$\cos\Theta = \cos\Theta_0 = \frac{\mathrm{RC}}{\mathrm{H}}. \tag{50}$$

The solution to Θ by the formula (50) also coincides with the classical solution for regular precession.
Let us now consider a solution in consideration of the angle of proper rotation Φ.

$$\tan\Phi = -\frac{\mathrm{a}_{23}}{\mathrm{a}_{11}} = \frac{-\mathrm{a}_{21}^1 \sin\Theta_0 + \mathrm{a}_{23}^1 \cos\Theta_0}{-\mathrm{a}_{11}^1 \sin\Theta_0 + \mathrm{a}_{13}^1 \cos\Theta_0} \tag{51}$$

After calculations we have:

$$\tan\Phi *= -\tan\Omega t\ \Phi *= -\Omega t, \dot{\Phi} *= -\Omega. \tag{52}$$

The obtained formulas coincide with the formulas of the classical solution, but with zero initial angles of precession and proper rotation.

Let us now consider a variant of the solution to the problem for irregular precession.

For the initial angles $\Phi_0 = \Psi_0 = 0; tan\Theta_0 = \frac{Aa}{CR}$ that differ from the angles (45), which generate regular precession, only by the sign of the angle of nutation. After transformations, the formulas for determining the Euler angles for the SEG are:

$$\tan\Psi^* = -\frac{\sin n\,t}{2\cos^2\Theta_0 - \cos 2\,\Theta_0 \cos n\,t}$$

$$\cos\Theta^* = \frac{\cos\Theta_0 \cos 2\,\Theta_0}{\tan^2\Theta_0} + 2\sin^2\Theta_0 \cos\Theta_0 \cos n\,t$$

$$\tan\Phi^* = \frac{\sin\,\Theta_0 \cos\,2\Theta_0 \sin\,\Omega t + \sin\,2\Theta_0 \sin\,nt \cos\,\Omega t}{\sin\,\Theta_0 \cos\,2\Theta_0 \cos\,\Omega t - 2\sin\,2\Theta_0 \cos\,nt\cos\,\Omega t} \tag{53}$$

The expressions (53) suggest that only the change of the sign of the initial angle of nutation – with the other two initial angles unchanged – caused the appearance of irregular precession motions in the Euler gyroscope.

IV. Mathematical Modeling

Figures 3 – 8 show the results of mathematical modeling using the kinematic Euler equations, which confirm the obtained analytical results.

Figures 3 and 4 present graphs of the modeling process for the Euler Ψ,Θ,Φ and the Euler-Krylov angles change, respectively, for the initial angles

$$\Theta(0) = \Theta_0; \theta(0) = \theta_0 = \Theta_0; \Psi_0 = \Phi_0 = \psi_0 = \varphi_0 = 0, \tag{M.1}$$

That is, corresponding to the conditions (23) of regular precession in the Euler angles. The relationship between the Euler and the Euler-Krylov angles is established due to the equality of the respective elements of the matrices (7) and (8).

SEG parameters

$$A = 0.1,\ sN\cdot cm\cdot s; s = 0.2,\ sN\cdot cm\cdot s;\quad a = 10^3, rad/s;\ R = 1570, rad/s;$$

$$\Omega = ({}^{c}/{}_{A} - 1)R = 1.57\cdot 10^3, rad/s \tag{M.2}$$

$$\Theta_0 = -\arctan\left(\frac{\mathrm{aA}}{\mathrm{Rc}}\right) = -0.308,\ \mathrm{rad.}$$

The graphs in Fig. 3 depict the change of the Euler angles for regular precession. The same cannot be said about the graphs in Fig. 4 for the Euler-Krylov angles – where one can see harmonic oscillations for the angles ψ and θ with a frequency slightly higher than 500 Hz, and for the angle φ, its increscent property is evident.

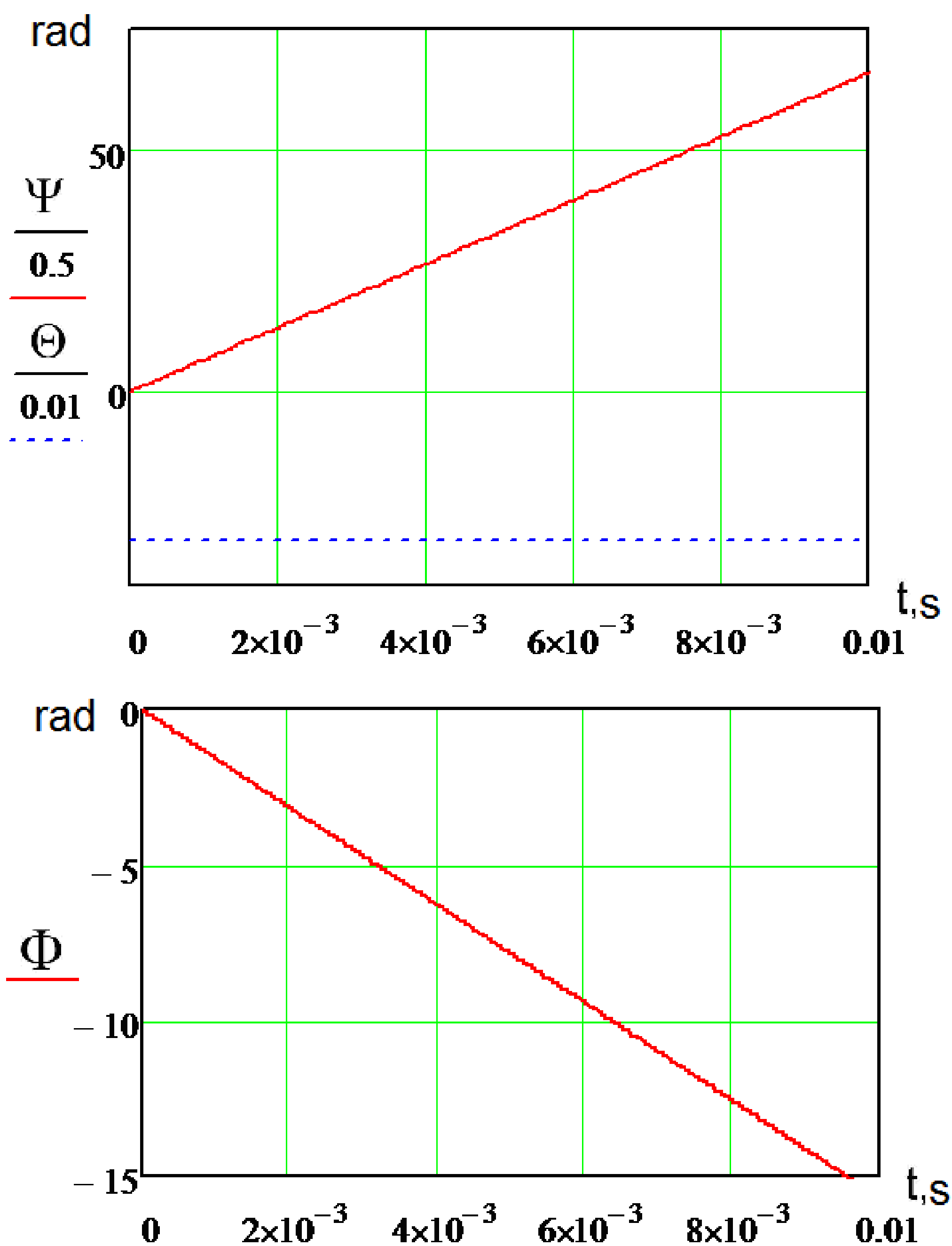
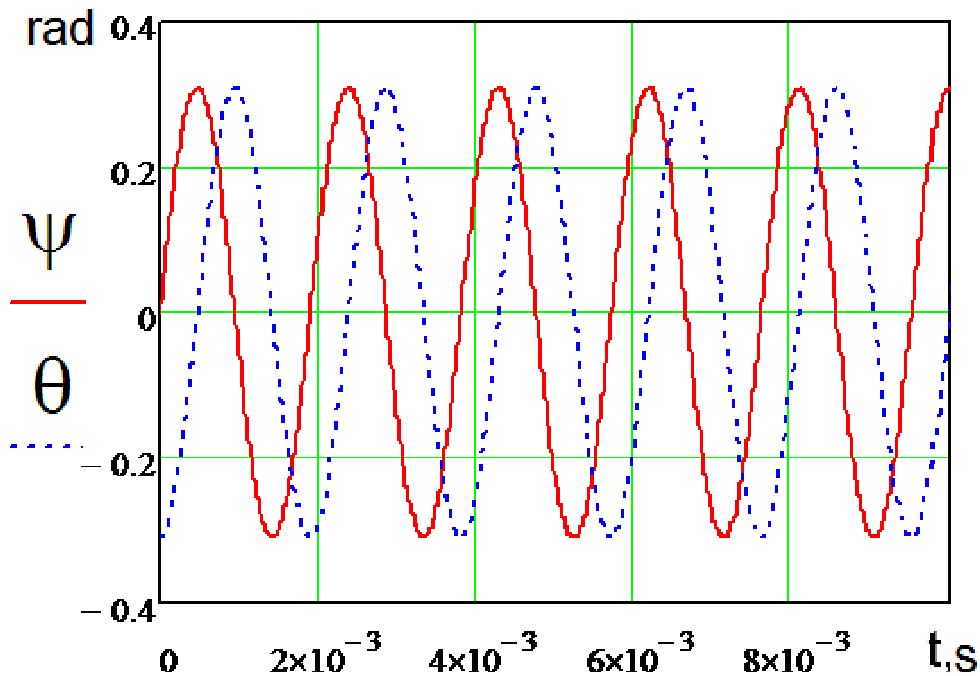

rad
50
Ψ
0.5
Θ
0.01
0
t,s
0
2×10⁻³
4×10⁻³
6×10⁻³
8×10⁻³
0.01
rad
0
– 5
Φ
– 10
– 15
t,s
0
2×10⁻³
4×10⁻³
6×10⁻³
8×10⁻³
0.01

Fig. 3



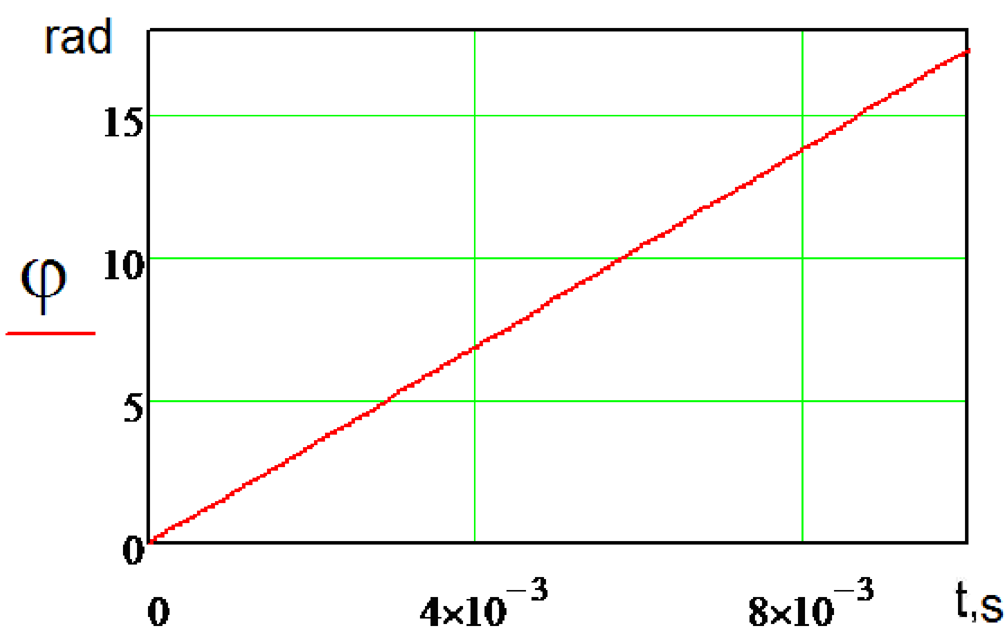


Fig. 4

When applying a stronger rotational pulse around the axis Ox for which $a = 4000, rad/s > R$, with unchanged other conditions for Fig. 3 and 4, the nature of the motion does not change (therefore, the graphs are not shown), however, for the Euler angles we have:

$$\Psi_{\max}(0.01) \cong 50\text{rad}, \Theta = \Theta_0 = -0.905\text{rad} = \text{const}, \Phi_{\max}(0.01) = -15.7\text{rad}.$$

For the Euler-Krylov angles in Fig. 6, the oscillation amplitudes along ψ and θ are equal to 0.905 rad, the frequencies are approximately equal to 870 Hz. The angle φ is increscent with superimposed frequency fluctuations of 1740 Hz.

Additionally, with unchanged parameters of modeling of SEG motions according to (M.1), (M.2) (figures 3 and 4), but with the sign of the initial angles of nutation reversed and equal to $\Theta_0 = \theta_0 = 0.308 rad$, the motion patterns shown in figures 5 and 6 were obtained.

In Fig. 5, for the Euler angles, the motion has acquired the character of irregular precession, namely, along Ψ and Θ - a vibrational pattern with frequencies slightly above 500 Hz of different amplitudes with oscillation centers shifted by about 0.3 rad. For the angle Φ, the velocity sign in Fig. 3 has changed to the opposite, and the angle become increscent. The graphs confirm the derived formulas (30).

For the Euler - Krylov angles, the motion is of a qualitatively similar character.

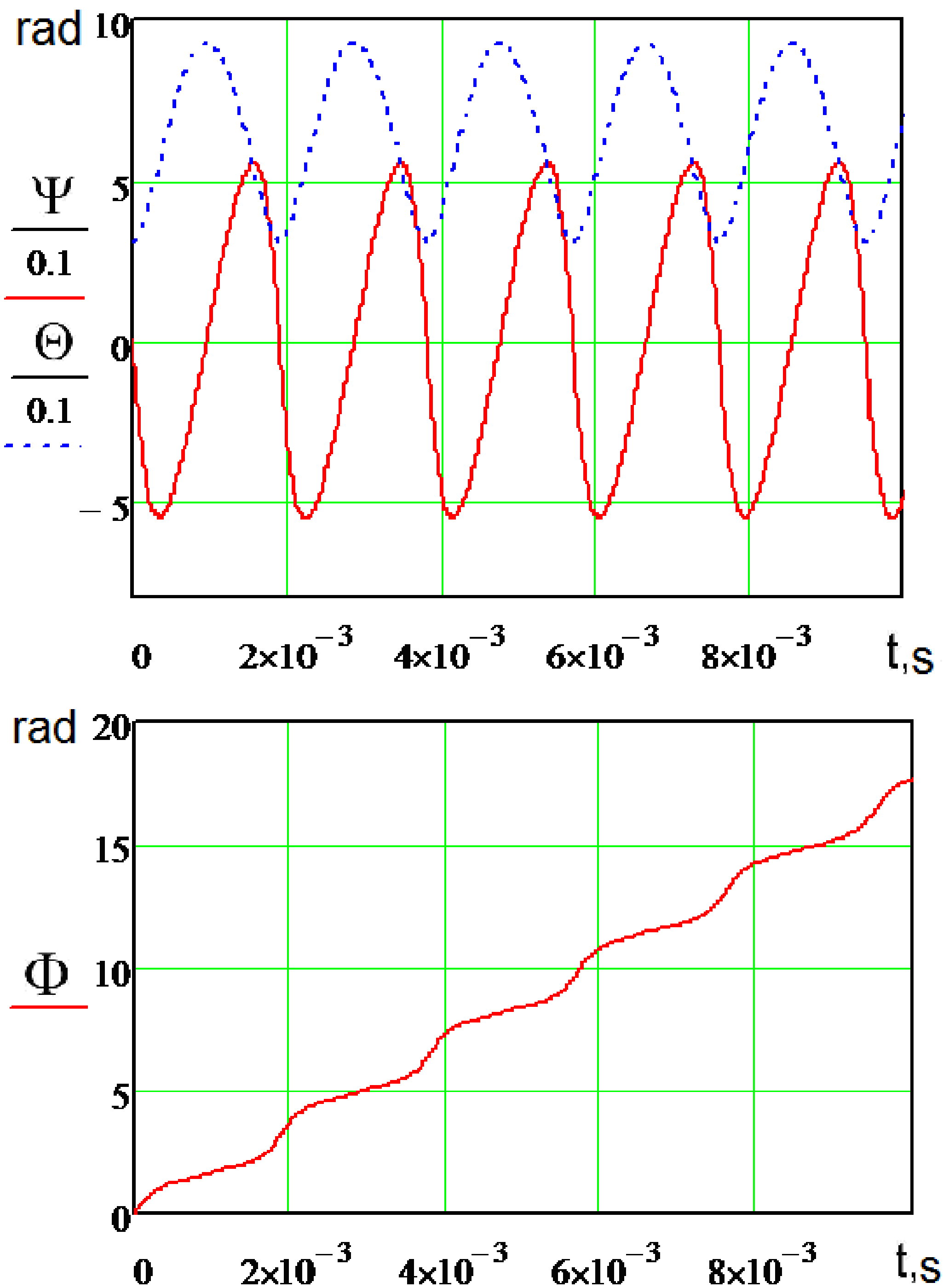


Fig. 5

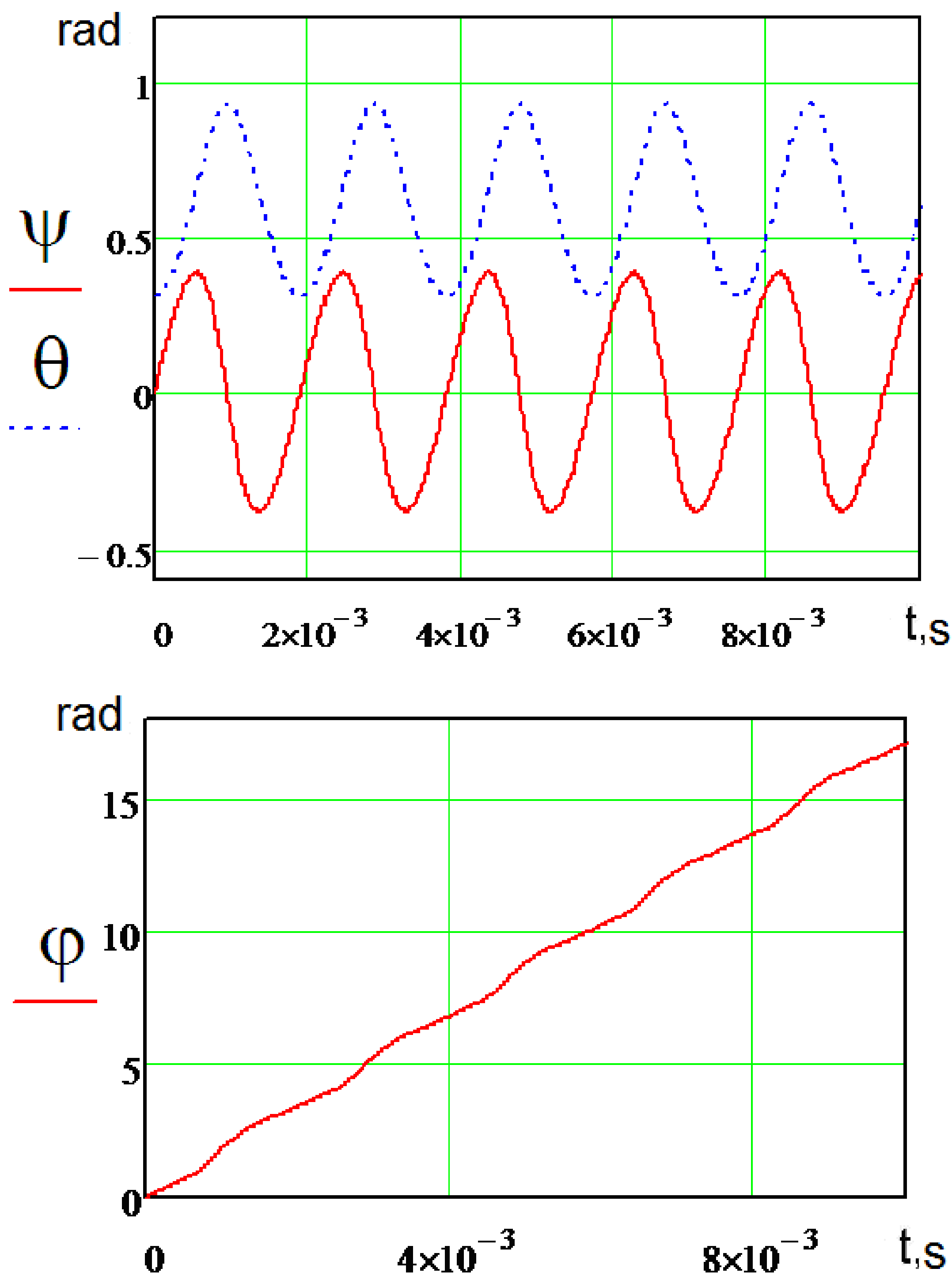
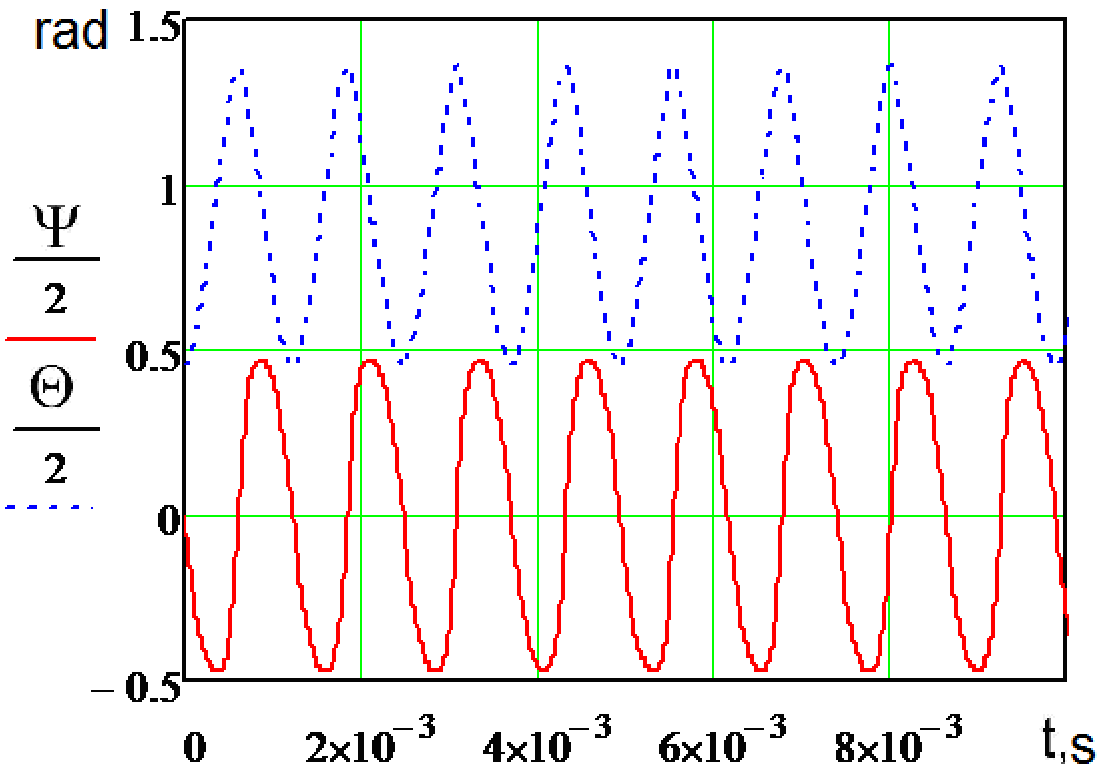

rad
1
0.5
0
−0.5
ψ
θ
0
2×10⁻³
4×10⁻³
6×10⁻³
8×10⁻³
t,s
rad
15
10
5
0
φ
0
4×10⁻³
8×10⁻³
t,s

Fig. 6


rad
1.5
1
0.5
0
– 0.5
Ψ
2
Θ
2
0
2×10^−3
4×10^−3
6×10^−3
8×10^−3
t,s

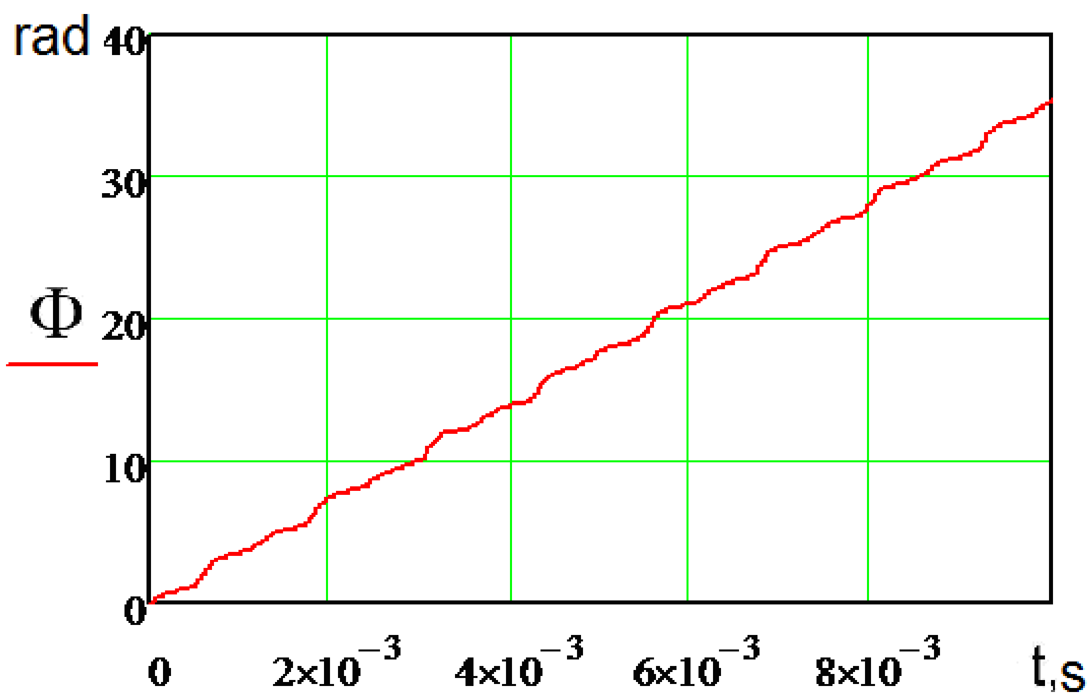

rad
40
30
20
10
0
Φ
0
2×10^−3
4×10^−3
6×10^−3
8×10^−3
t,s

Fig. 7

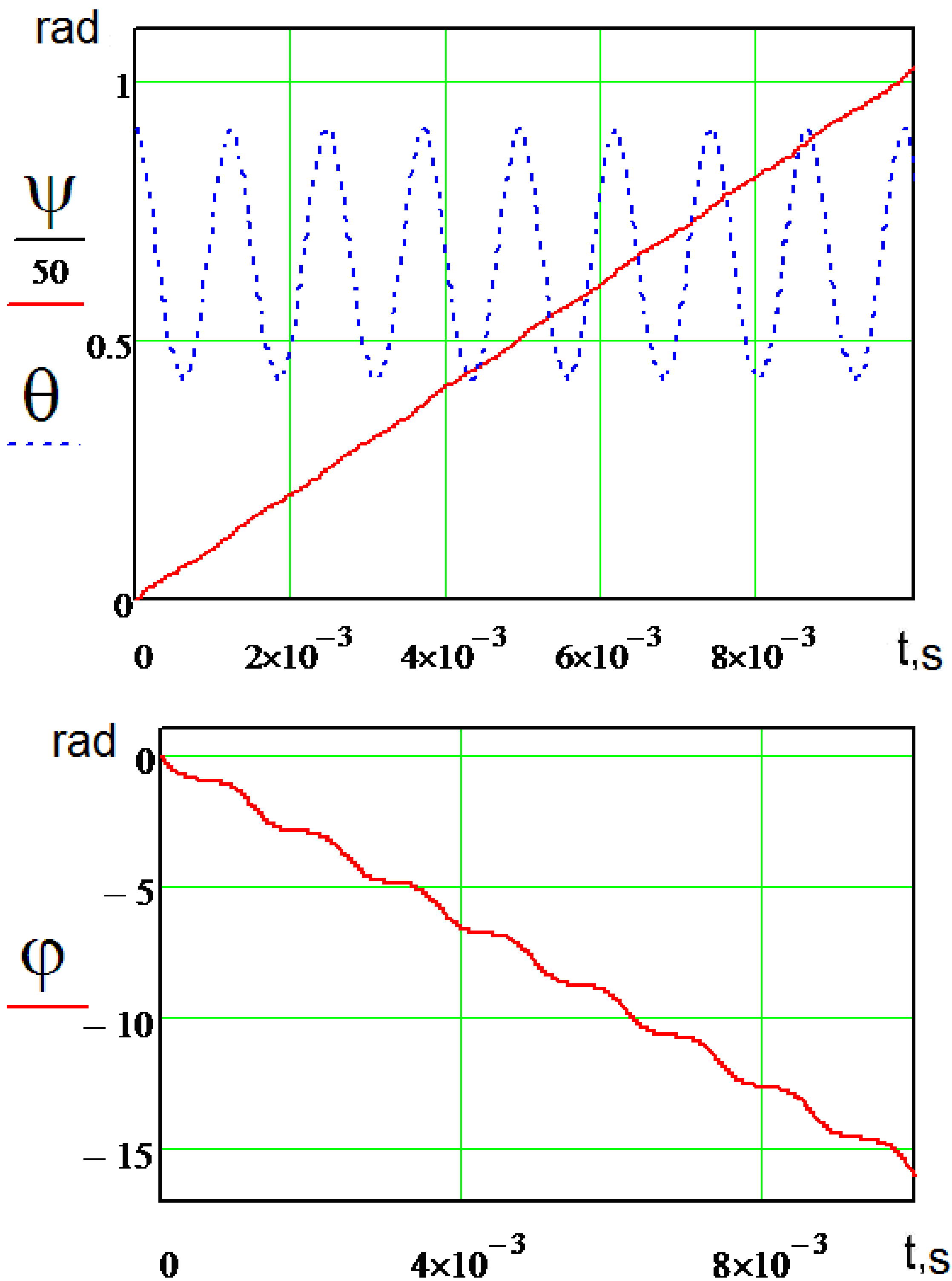


Fig. 8

Figures 7 and 8 show the results of modeling of the SEG parameters and motions that correspond to figures 5 and 6 with the only difference: angular velocity is provided equal to $a = 4000, rad/s$, $a > R$. As the result, the nature of motions along the Euler angles (Fig. 7) did not change qualitatively, while quantitatively, the vibration centers moved apart to the angles Ψ and to Θ up to 0.45 rad, and the oscillation frequencies increased up to 820 Hz. The angle Φ remains to be in crescent with superimposed oscillations.

At the same time, the motion for the Euler-Krylov angles has changed dramatically (Fig. 8). The angle Ψ began to increase monotonically in the direction of the rotational pulse action, which is novel. The angle θ is still oscillatory in nature with a frequency of 820 Hz around the shifted center of oscillations, and the angle φ has changed the sign to the opposite in relation to Fig. 6.

V. Conclusion

According to the results of mathematical modeling, it is shown that the motions that correspond to regular precession in the Euler angles are independent of the magnitude of the angular velocity a, which is caused by the action of the rotational pulse. However, a change of the sign of the initial angle of nutation leads to a sharp change in the nature of motion — it becomes irregular, which is reflected in the explanation for Fig. 5. The motion along the Euler-Krylov angles radically depends on a: with $a > R$, the angle ψ becomes monotonically increscent in the direction of the pulse action, and the angle of proper rotation changes the sign of its monotonic rotation to the opposite. Additionally, in the article:

- It was proven that regular precession in SEG is possible only for the initial Euler angles determined by the known formulas:

$$\psi_0 = const; \quad \varphi_0 = 0; \qquad \cos\theta_0 = const = {Cr_0}/{G}.$$

For any other initial angles regular precession is not possible.

- An analytical solution to the problem of the SEG motion was found by integrating the matrix differential quaternion kinematic equations, as well as the Poisson equations. Formulas for determining the Euler and the Euler-Krylov angles were derived.
- The obtained formulas and mathematical modeling confirmed that for the angles, that are different from the initial Euler angles (1), precession that is different from the regular one is present in SEG.

As for corpuscular gyroscopes, based on this study, it can be assumed that depending on the application of an external magnetic field over time, not only Larmor precession [14], but also "pseudo-Larmor" precession is possible in them.

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