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I. Introduction

In aerodynamics the air flows over (aircraft wing) or inside (pipes and nozzles) solid bodies. Low speed flows are usually treated as 'incompressible', that is, changes in density of air as it flows are ignored (density is assumed constant). However, that is only an assumption and not a reality; in fact Anderson[1] calls it a myth. The analysis of flows as incompressible will always be in error, howsoever small, but the results may be in error by a small amount which may be acceptable in practical life. It would be preferable if the analysis can be carried out as compressible flow provided the mathematics is not too involved. That is an aim of this communication.

II. Analysis of Normal Shock Wave

Fig, 1 shows the sketch of a normal shock wave. V_1, P_1, T_1 and ρ_1 are the velocity, pressure, temperature and density of air just before the shock wave. V_2, P_2, T_2 and ρ_2 are the parameters just after the shock wave.

The parameters are related to each other by the following equations:

Continuity: $\rho_1.V_1 = \rho_2.V_2$ (1)

Momentum: $P_1 + \rho_1.V_1^2 = P_2 + \rho_2.V_2^2$ (2)

Gas law: $P_1 = \rho_1.R.T_1$ and $P_2 = \rho_2.R.T_2$ (3)

Where R is the gas constant. R = 287 J/kg.K for air.

Energy equation:

$$c_p.T_1 + \frac{{V_1}^2}{2} = c_p.T_2 + \frac{V_2^2}{2} \quad (4)$$

Dividing eqn 2 by eqn 1:

$$\frac{P_1}{\rho_1 V_1} + V_1 = \frac{P_2}{\rho_2 V_2} + V_2 \quad (5)$$

Using gas law (eqn.3), eqn 5 can be simplified to:

$$R.\left(\frac{T_1}{V_1} - \frac{T_2}{V_2}\right) = V_2 - V_1 \quad (6)$$

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Or

$$\frac{T_2}{{V_1}^2} = T_1\frac{V_2}{{V_1}^3} - \left(\frac{V_2}{V_1}\right)^2\frac{1}{R} + \left(\frac{V_2}{V_1}\right)\frac{1}{R} \quad (7)$$

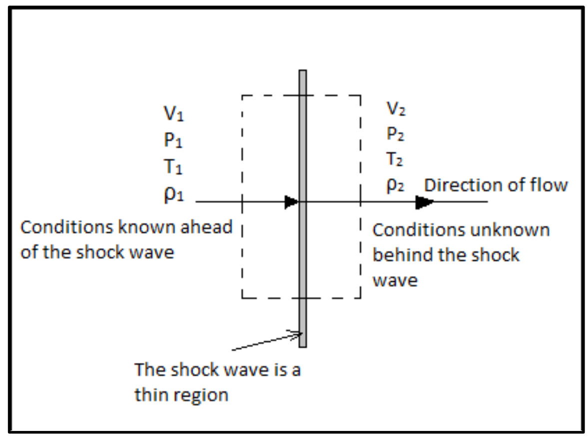

Fig. 1: Sketch of a normal shock wave

Knowing that $c_p = \frac{\gamma R}{\gamma - 1}$ energy equation (4) can be simplified to:

$$\left(\frac{V_2}{V_1}\right)^2 + \frac{2\gamma R}{\gamma-1}\frac{T_2}{{V_1}^2} = 1 + \frac{2\gamma R}{\gamma-1}\frac{T_1}{{V_1}^2} \quad (8)$$

Substituting T2 from eqn 7 into eqn 8 and simplifying:

$$\left(\frac{V_2}{V_1}\right)^2\left(1 - \frac{Z}{R}\right) + ZV_2\left(\frac{T_1}{{V_1}^3} + \frac{1}{V_1 R}\right) = 1 + \frac{ZT_1}{{V_1}^2} \quad (9)$$

where $Z = \frac{2\gamma R}{\gamma - 1}$.

It may be noted that V_2 can be determined from eqn 9 since all other parameters pertain to upstream of the shock wave and are known.

With V_2 known, T_2 can be calculated from eqn 7. Eqn 5 can be simplified with the help of eqn 1:

$$P_2 = P_1 + \rho_1{V_1}^2 - \rho_1 V_1 V_2 \quad (10)$$

Downstream pressure P_2 can be calculated from eqn 10.

Downstream density ρ_2 can be calculated from eqn 3 as all parameters are known.

III. Solution of Equations

V_2 can be determined from eqn 9. It is, however, not easy to solve the equation. The 'goal seek' facility of spreadsheet (like Excel of Microsoft Office) is used for this purpose. Once V_2 is known, temperature, pressure and density downstream of the shockwave are easily calculated.

It can thus be seen that it is not necessary to read Mach number from any table for solving the normal shock wave problem. The sonic velocity and Mach number at both upstream and downstream regions of the shock wave can be easily calculated.

Table 1 gives one example of solution of a normal shock wave problem.

	A	B	C	D
1		*Table 1:* Solution of normal shock wave problem		
2			Case 1	Case 2
3		ϒ	1.4	1.4
4		R, J/kg.K	287	287
5		V1, m/s	680	680
6		T1, K	288	288
7		P1,Pa	101320	101320
8		P1,atm	1	1
9		Ro1, kg/m3	1.226	1.226
10		Z	2009.000	2009.000
11		RHS	2.251	2.251
12		a	-6.000	-6.000
13		b	0.012134236	0.012134236
14		V2	680.042	255.139
15		LHS	2.251	2.251
16		c	288.018	108.059
17		d	1611.351	226.816
18		e	1611.250	604.512
19		T2, K	287.917	485.755
20		P2, Pa	101284.592	455460.903
21		P2, atm	1.000	4.495
22		Ro2, kg/m3	1.226	3.267
23		c1	340.174	340.174
24		c2	340.125	441.788
25		M1	1.999	1.999
26		M2	1.999	0.578
27		P01, atm	3.797	3.797
28		P02, atm	3.797	5.545

Procedure for arriving at the solutions is described below.

In this example V_1 = 680 m/s which means the flow is supersonic before the shock wave. The flow will then be subsonic after the shock wave. This means there will be two solutions of the equations: one where there is no shock and the second where a shock wave occurs. To arrive at the first solution (case 1), assume an arbitrary value of V_2 greater than 680 m/s, say 800 m/s. In this case LHS is 1. 403 (cell C15). Click on Data and then What-if Analysis in the spreadsheet. Then select Goal Seek. In Set Cell enter C15, the cell that contains value of LHS. In 'To value' enter 2.251, the value of RHS in cell C11. In 'By changing cell' enter C14, the cell that contains the value of V_2. Click OK. The cell C14 will now show value close to 680 which is the same as value of V_1. This means that no shock has occurred in this case. This shows that the Excel has been properly filled. Click OK. The values of T_2, P_2, ρ_2, c_2, M_2 and $P0_2$ in cells C19, C20, C22, C24, C26 and C28 respectively will automatically update.

If a shock has occurred then, downstream of the shock wave, the velocity will be subsonic. Refer to values in column D (case 2). Enter a small value, say 50, in cell D1. Values in subsequent cells will change. Use Data-What if analysis-Goal Seek as before and enter D15 in Set Cell, 2.251 in 'To Value' and D14 in 'By changing cell' and click OK. The value of V_2 in cell D14 is now 255.139 m/s. The values of T_2, P_2, ρ_2, c_2, M_2 and $P0_2$ are automatically updated. It can be noted that P_2 = 4.495 atm and M_2 = 0.578. A complete solution of the problem is thus in the Spreadsheet.

It can be observed that there was no need to read Mach number value from charts; the problem could be easily solved in a normal mathematical fashion. However, if desired values of sonic velocity and mach number before and after the shock could be easily calculated as shown in Table 1. Even total pressure values upstream and downstream of the normal shock could be calculated.

IV. Pitot Tube Analysis

Pitot static tubes are used to measure flow velocities. There are three cases, viz. incompressible subsonic, compressible subsonic and compressible supersonic. These cases are analyzed below.

a) *Incompressible subsonic flow*

The instrument used to measure velocity of fluid at any location is called Pitot-static tube. The Pitot has two openings, one which faces opposite to the flow direction and another which is perpendicular to the flow direction. The former measures the total or stagnation pressure at the location of the Pitot tube and the latter measures the static pressure. No flow takes place through any of these two openings. The stagnation pressure represents a sum of the static pressure and the dynamic pressure. The difference between the stagnation pressure and the static pressure is measured by appropriate means. Bernoulli's equation is applied to the two readings. The stagnation pressure P0 is related to the static pressure P1 by the equation:

$$P_1 + \frac{1}{2}\rho V_1{}^2 = P_0 \qquad (11)$$

from which velocity V_1 is determined as:

$$V_1 = \sqrt{\frac{2(P_0 - P_1)}{\rho}} \qquad (12)$$

As an example, if P_0 = 104326 Pa, P_1 = 101325 Pa and ρ = 1.223 kg/m^3,

$$V_1 = \sqrt{\frac{2(104326-101325)}{1.225}} = 70\ m/s \qquad (13)$$

b) Compressible subsonic flow

The Pitot tube can also be used to determine flow velocity in high speed but subsonic flow. The formula for velocity V1 in a compressible subsonic flow is:

$$V_1 = \left[\frac{2\gamma P_1}{\rho(\gamma-1)}\left\{\left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right\}\right]^{\frac{1}{2}} \qquad (14)$$

For the case with P0 = 1104326 Pa, P1 = 101325 Pa, ϒ = 1.4 and ρ = 1.225 kg/m3,

V_1 = 69.61 m/s. It may be noted that the difference between incompressible velocity and compressible velocity is hardly 0.39 m/s or 0.56%.

In case P_0 = 140000 Pa, the incompressible velocity will be 251.28 m/s whereas compressible velocity will be 236.63 m/s. The difference is 14.65 m/s or 6.19% which may be difficult to ignore.

c) Supersonic flow

When a Pitot tube is used to measure flow velocity in supersonic flow, it acts as an obstruction to the flow and velocity is brought down to zero in front of it. But transition from supersonic flow to subsonic flow means a shock has occurred somewhere on the way. It is apparent that the Pitot tube will measure total pressure behind the shock; hence it is necessary to determine relationship between dynamic or total pressure behind the shock and flow velocity before the shock. Table 2 shows the calculation.

Column C gives data for a random case in supersonic flow. In the case shown, flow velocity of 350 m/s (cell C7) will give rise to Pitot tube reading of 198845 Pa (cell C26). Consider a Pitot tube reading of 275000 Pa for which we need to determine flow velocity. A two or three step iteration will be needed.

	A	B	C	D	E	F
1						
2		*Table 2:* Supersonic Pitot Tube				
3				Step 1	Step 2	Step 3
4		cp, J/kg.K	1004.5	1004.5	1004.5	1004.5
5		ϒ	1.4	1.4	1.4	1.4
6		R, J/kg.K	287	287	287	287
7		V1, m/s	350.0	446.8	441.3	441.5
8		T1, K	287	287	287	287
9		P1,Pa	101320	101320	101320	101320
10		P1,atm	1	1	1	1
11		Ro1, kg/m3	1.230	1.230	1.230	1.230
12		Z	2009	2009	2009	2009
13		RHS	5.707	3.889	3.961	3.958
14		a	-6.000	-6.000	-6.000	-6.000
15		b	0.033448	0.022135	0.022572	0.022552
16		**V2, m/s**	**332.9**	**289.7**	**291.4**	**291.4**
17		LHS	5.707	3.890	3.961	3.958
18		c	272.959	186.133	189.493	189.387
19		d	386.088	292.512	295.794	295.794
20		e	405.948	451.028	448.000	448.252
21		**T2, K**	**292.8**	**344.6**	**341.7**	**341.8**
22		**P2, Pa**	**108692**	**187607**	**182703**	**182884**
23		P2, atm	1.073	1.852	1.803	1.805
24		**Ro2, kg/m3**	**1.293**	**1.897**	**1.863**	**1.864**
25		**T02, K**	**348.0**	**386.4**	**384.0**	**384.1**
26		**P02, Pa**	**198845**	280029	274774	275000
27		**P02,atm**	**1.962**	**2.764**	**2.712**	**2.714**
28		c1, m/s	339.6	339.6	339.6	339.6
29		c2, m/s	343.0	372.1	370.5	370.6
30		M1	1.031	1.316	1.300	1.300
31		M2	0.970	0.779	0.786	0.786

Under Data-What if Analysis-Goal seek, 'Set Cell' to C26, 'To Value' to 275000, 'By changing cell' to C7, and enter. The values in the column C will change. Use Data-What if Analysis-Goal seek again and ensure that cell C17 reads 3.889 (same as in cell C13) and cell C16 reads 289.7 m/s. (Column D in Table 2 depicts values that one will see in column C). The value in column C26 will now read 280029 which is somewhat different from the desired value of 275000. Repeat the iteration and we get the value of 274774 in cell C26 (as depicted in cell E26 in Table 2). One more iteration gives value of 275000 (the desired figure) in cell C26 and the corresponding figure of velocity V_1 (441.5 m/s) in cell C7 (as depicted in cell F7 in Table 2). Thus a Pitot tube reading of 275000 Pa (equivalent to 2.714 atm) corresponds to V_1 = 441.5 m/s. Sonic velocity upstream of shock is calculated as 339.6 m/s and downstream of shock as 370.6 m/s. Mach numbers upstream and downstream of shock are 1.300 and 0.786 respectively.

Another example is shown in Table 3. It can be seen that Pitot tube reading of 1221980 Pa (12.06 atm) represents a velocity upstream of shock as 1018.7 m/s. Sonic velocity upstream of shock is calculated as 339.6 m/s and downstream of shock as 555.8 m/s. Mach numbers upstream and downstream of shock are 3.0 and 0.475 respectively. Complete information about the example like downstream velocity, pressure, temperature and density is available in the spreadsheet. There never was a need to refer to any tables.

	A	B	C	D	E	F
1		*Table 3:* Supersonic Pitot Tube				
2				Step 1	Step 2	Step 3
3		cp, J/kg.K	1004.5	1004.5	1004.5	1004.5
4		γ	1.4	1.4	1.4	1.4
5		R, J/kg.K	287	287	287	287
6		V1, m/s	350.0	1034.3	1019.0	1018.7
7		T1, K	287	287	287	287
8		P1,Pa	101320	101320	101320	101320
9		P1,atm	1	1	1	1
10		Ro1, kg/m3	1.230	1.230	1.230	1.230
11		Z	2009	2009	2009	2009
12		RHS	5.707	1.539	1.555	1.556
13		a	-6.000	-6.000	-6.000	-6.000
14		b	0.033448	0.007289	0.007414	0.007417
15		V2, m/s	332.9	265.4	264.1	264.1
16		LHS	5.707	1.540	1.555	1.555
17		c	272.959	73.649	74.377	74.400
18		d	386.088	245.443	243.002	243.002
19		e	405.948	956.451	937.674	937.393
20		T2, K	292.8	784.7	769.0	768.8
21		P2, Pa	108692	1079458	1047631	1046964
22		P2, atm	1.073	10.654	10.340	10.333
23		Ro2, kg/m3	1.293	4.793	4.746	4.745
24		T02, K	348.0	819.7	803.8	803.5
25		P02, Pa	198845	1257929	1222696	1221980
26		P02,atm	1.962	12.415	12.067	12.060
27		c1, m/s	339.6	339.6	339.6	339.6
28		c2, m/s	343.0	561.5	555.9	555.8
29		M1	1.031	3.046	3.001	3.000
30		M2	0.970	0.473	0.475	0.475

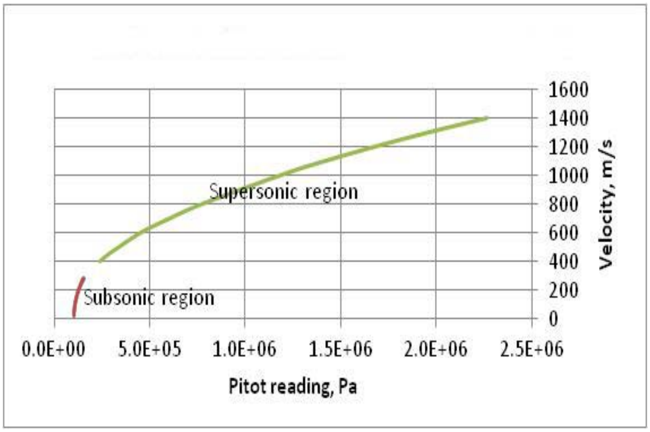

Fig. 2: Flow velocity as a function of Pitot tube reading

Fig. 2 shows the variation of flow velocity as a function of Pitot tube reading for both subsonic and supersonic regions.

V. Flow Through Nozzles

The spreadsheet solution of compressible flow is covered in literature[2]. What follows is an improved version of the same. Consider a convergent-divergent nozzle as shown in fig. 3.

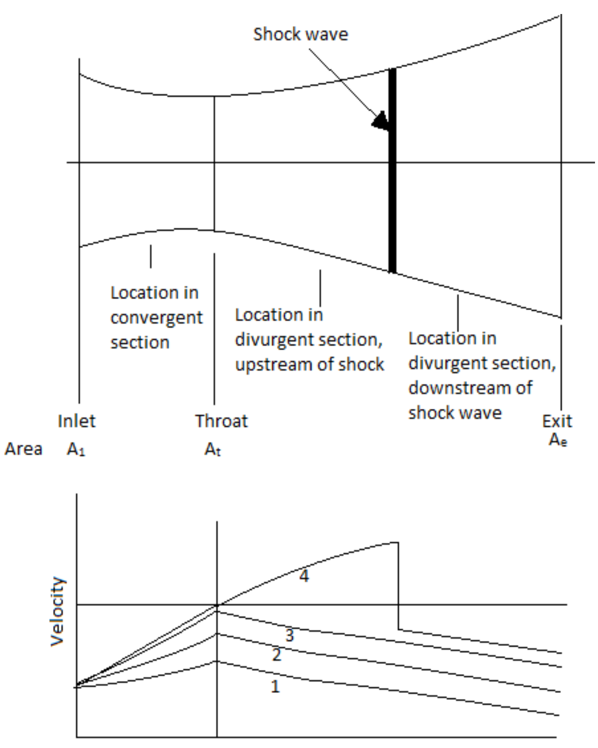

Fig. 3: A typical convergent-divurgent nozzle. Curves 1, 2, 3 and 4 are for different pressure values at exit

Equation of state: $p = \rho RT$ where ρ is density of the fluid, p is the absolute pressure and T is the absolute temperature. R is the gas constant.

Continuity equation: $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$ or $\rho_1/\rho_2) = (A_2/A_1)(V_2/V_1)$ (A_1 and A_2 are cross sectional areas of two locations in the nozzle and V_1 and V_2 are velocities at these locations respectively. Location 2 could be anywhere in the nozzle between inlet and exit).

From equation of state:

$$T_2 = (p_2/p_1)(A_2/A_1)(V_2/V_1).T_1 \tag{15}$$

Steady state energy equation (neglecting potential energy):

$$\frac{{V_2}^2}{2} + \frac{\gamma}{\gamma-1}RT_2 = \frac{{V_1}^2}{2} + \frac{\gamma}{\gamma-1}RT_1$$

Or

$${V_2}^2 + Z.T_2 = {V_1}^2 + Z.T_1 \tag{16}$$

where $Z = 2\gamma R/(\gamma-1) = 2009$ for air.

For isentropic flow: $\frac{p_1}{{\rho_1}^\gamma} = \frac{p_2}{{\rho_2}^\gamma}$

Or

$$\left(\frac{p_1}{p_2}\right) = \left({}^{A_2}/{}_{A_1}\right)^\gamma \left({}^{V_2}/{}_{V_1}\right)^\gamma \tag{17}$$

From equations 1, 2 and 3,

$$\left(\frac{V_2}{V_1}\right)^2 + Z.\left(\frac{V_2}{V_1}\right)^{(1-\gamma)}\left(\frac{A_1}{A_2}\right)^{(\gamma-1)}\frac{T_1}{{V_1}^2} = 1 + \frac{Z.T_1}{{V_1}^2} \tag{18}$$

Velocity V_2 at the downstream location can be calculated from eqn (18) for any value of A_2 since all other information pertains to upstream and is known.

	A	B	C	D
1		*Table 4:* Nozzle flow analysis		
2				
3		cp	1004.5	1004.5
4		γ	1.4	1.4
5		R	287	287
6		A1	0.007854	0.00785398
7		P1, atm	1	1
8		P1, Pa	1.01E+05	1.01E+05
9		T1, K	313	313
10		V1, m/s	100	100
11		Ro1, kg/m3	1.13E+00	1.13E+00
12		Z	2009	2009
13		RHS	63.8817	63.8817
14		V2, m/s	99.999787	258.772587
15		V2/V1	0.9999979	2.58772587
16		A2	0.007854	0.00384845
17		A1/A2	0.9999977	2.04081633
18		LHS	63.88169	63.8808525
19		P2, Pa	1.01E+05	7.27E+04
20		P2, atm	1.00E+00	7.17E-01
21		P1/P2	1.00E+00	1.39E+00
22		T2, K	312.99997	284.641749
23		T1/T2	1.0000001	1.09962787
24		Ro2, kg/m3	1.13E+00	8.90E-01
25		Ro1/Ro2	1.00E+00	1.27E+00
26		c1, m/s	354.63136	354.631358
27		c2, m/s	354.63134	338.184941
28		M1	0.281983	0.28198296
29		M2	0.2819824	0.76518069

Pressure P_2 at the downstream location can be calculated from eqn (17) and temperature T_2 can be calculated from eqn (15). All the information is available to determine the density at the downstream location. Table 4 is a sample of the nozzle flow analysis. It may be noted that column C is for parameters at the first section and column D is for second section where areas A_1 = 0.00785 sq m and A_2 = 0.00385 sq m. Also this table represents data for the subsonic solution of the problem. A second solution exists for the supersonic section of the problem. To arrive at the supersonic section of the solution, give a large value (say 1000) of V_2 (row 14) as the initial guess of V_2 and solve the problem through 'What-if-Analysis'.

VI. Conclusion

It is shown that compressible flow can be analyzed without going through the step of referring to pre-calculated tables. Use of spreadsheet makes it possible. In general, the reading of Mach number M may lie between two values and interpolation is required; the interpolated value could be in error. Also the complete problem is solved in one step, i.e. all parameters are determined in one step.

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