New Types of Transitive Maps and Minimal Mappings

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Abstract - In this paper, we have introduced the relationship between two different concepts of maps, namely topological $\alpha$-transitive and $\delta$-transitive maps and investigate some of their properties in two topological spaces $(X, \tau^\alpha)$ and $(X, \tau^\delta)$. $\tau^\alpha$ denotes the $\alpha$-topology and $\tau^\delta$ denotes the $\delta$-topology of a given topological space $(X, \tau)$. The two concepts are defined by using the concepts of $\alpha$- irresolute and $\delta$- irresolute maps respectively. Also, we studied the relationship between two types of minimal systems, namely, $\alpha$- minimal and $\delta$- minimal systems, The main results are the following propositions.

Keywords: topologically $\delta$-transitive, $\alpha$- irresolute, $\delta$- transitive, $\delta$- dense.

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New Types of Transitive Maps and Minimal Mappings

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Abstract- In this paper, we have introduced the relationship between two different concepts of maps, namely topological $\alpha$-transitive and $\delta$-transitive maps and investigate some of their properties in two topological spaces $(X, \tau^\alpha)$ and $(X, \tau^\delta)$. $\tau^\alpha$ denotes the $\alpha$-topology and $\tau^\delta$ denotes the $\delta$-topology of a given topological space $(X, \tau)$. The two concepts are defined by using the concepts of $\alpha$-irresolute and $\delta$-irresolute maps respectively. Also, we studied the relationship between two types of minimal systems, namely, $\alpha$-minimal and $\delta$-minimal systems. The main results are the following propositions:

1. Every topologically $\alpha$-transitive map implies topologically $\delta$-transitive map, but the converse not necessarily true.
2. Every $\alpha$-minimal system implies $\delta$-minimal system, but the converse not necessarily true.

Keywords: topologically $\delta$-transitive, $\alpha$-irresolute, $\delta$-transitive, $\delta$-dense.

I. Introduction

Let $(X, \tau)$ be a topological space, $f: X \to X$ be $\alpha$-irresolute map, then the set $A \subseteq X$ is called topologically $\alpha$-mixing set[1] if, given any nonempty $\alpha$-open subsets $U, V \subseteq X$ with $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$ then $\exists N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$, weakly $\alpha$-mixing set[4] of $(X, f)$ if any choice of nonempty $\alpha$-open subsets $V_1, V_2$ of $A$ and nonempty $\alpha$-opensubsets $U_1, U_2$ of $X$ with $A \cap U_1 \neq \emptyset$ and $A \cap U_2 \neq \emptyset$ there exists $n \in \mathbb{N}$ such that $f^n(V_1) \cap U_1 \neq \emptyset$ and $f^n(V_2) \cap U_2 \neq \emptyset$, strongly $\alpha$-mixing if for any pair of open sets $U$ and $V$ with $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$, there exist some $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for any $k \geq n$. A point $x$ which has $\alpha$-dense orbit $O_{\alpha}(x)$ in $X$ is called $\alpha$-type hyper-cyclic point. A system is $\alpha$-mixing[1] if, given $\alpha$-open sets $U$ and $V$ in $X$, there exists an integer $N$, such that, for all $n > N$, one has $f^n(U) \cap V \neq \emptyset$, topologically $\alpha$-mixing if for any nonempty $\alpha$-open set $U$, there exists $N \in \mathbb{N}$ such that $\bigcup_{n \geq N} f^n(U)$ is $\alpha$-dense in $X$. With the above concepts, some new theorems have been introduced and studied. Furthermore, we have the following results:

- Every topologically $\alpha$-transitive map implies topologically $\delta$-transitive map, but the converse not necessarily true.
- Every $\alpha$-minimal system implies $\delta$-minimal system, but the converse not necessarily true.
- $(E_a) \Rightarrow (ET_a)$;
- $(TM_a) \Rightarrow (WM_a) \Rightarrow (TT_a)$;

II. Preliminaries and Theorems

Definition 3.1 [2] A map $f: X \to Y$ is called $\alpha$-irresolute if for every $\alpha$-open set $H$ of $Y$, $f^{-1}(H)$ is $\alpha$-open in $X$.

Proposition 2.2 The product of two topologically $\alpha$-mixing systems must be topologically $\alpha$-mixing.

Proof: Suppose that $(X, f)$ and $(Y, g)$ are two $\alpha$-mixing systems, and consider any $\alpha$-open sets $W, W'$ in $X \times Y$. By definition of the product topology, there exist $\alpha$-open sets $U, U' \subseteq X$ and $V, V' \subseteq Y$ so that $U \times V \subseteq W$ and $U' \times V' \subseteq W'$. By definition of topological $\alpha$-mixing of $(X, f)$, there exists $N$ such that for any $n > N$, $f^n(U) \cap V \neq \emptyset$. By definition of topological $\alpha$-mixing of $(Y, g)$, there exists $N'$ such that for any $n > N'$, $g^n(V') \cap U \neq \emptyset$. Then, for any $n > \max(N, N')$, both $f^n(U) \cap V$ and $g^n(V') \cap U$ are nonempty, and therefore $(f \times g)^n(U \times U') \cap (V \times V')$ is nonempty as well. But this implies that $(f \times g)^n(W) \cap W' \neq \emptyset$, since $W$ and $W'$ were arbitrary, this implies that $(X \times Y, f \times g)$ is topologically $\alpha$-mixing.

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Theorem 2.3 The product of two $\alpha$-transitive maps is not necessarily $\alpha$-transitive map [4].

Corollary 2.4 The product of two topologically $\alpha$-transitive systems is not necessarily topologically $\alpha$-transitive.

III. NEW TYPES OF CHAOS OF TOPOLOGICAL SPACES

In this section, I introduced and defined $\alpha$-type transitive maps [3] and $\alpha$-type minimal maps [3], and study some of their properties and prove some results associated with these new definitions. I investigate some of their properties and prove some results.

Definition 3.1 Let $X$ be a separable and second category space with no isolated point, if for $x \in X$ the set $\{ f^n(x) : n \in \mathbb{N} \}$ is dense in $X$ then $x$ is called hyper-cyclic point. If there exists such an $x \in X$, then $f$ is called hyper-cyclic function or $f$ is said to have a hyper-cyclic point. Here, we have an important theorem that is: $f$ is a hyper-cyclic function if and only if $f$ is transitive.

Definition 3.2 A function $f: X \rightarrow X$ is called $\alpha r$-homeomorphism if $f$ is $\alpha r$-irresolute bijective and $f^{-1}: X \rightarrow X$ is $\alpha r$-irresolute.

Definition 3.3 Two topological systems $f: X \rightarrow X$, $g: Y \rightarrow Y$, $y_{n+1} = g(y_n)$ are topologically $\alpha r$-conjugate if there is $\alpha r$-homeomorphism $h: X \rightarrow Y$ such that $h \circ f = g \circ h$ (i.e. $h(f(x)) = g(h(x))$). We call $h$ a topological $\alpha r$-Conjugacy. Then I have proved some of the following statements:

1. The maps $f$ and $g$ have the same kind of dynamics.
2. If $x$ is a periodic point of the map $f$ with stable set $W_f(x)$, then the stable set of $h(x) = h(W_f(x))$.
3. The map $f$ is $\alpha r$-exact if and only if $g$ is $\alpha r$-exact.
4. The map $f$ is $\alpha r$-mixing if and only if $g$ is $\alpha r$-mixing.
5. The map $f$ is $\alpha$-type chaotic if and only if $g$ is $\alpha$-type chaotic.
6. The map $f$ is weakly $\alpha$-mixing if and only if $g$ is weakly $\alpha r$-mixing.

Remark 3.4
If $\{x_0, x_1, x_2, \ldots \}$ denotes an orbit of $x_{n+1} = f(x_n)$ then $\{ y_0 = h(x_0), y_1 = h(x_1), y_2 = h(x_2), \ldots \}$ yields an. In particular, $h$ maps periodic orbits of $f$ onto periodic orbits of $g$. orbit of $g$ since $y_{n+1} = h(x_{n+1}) = h(f(x_n)) = g(h(x_n)) = g(y_n)$, i.e. $f$ and $g$ have the same kind of dynamics.

I introduced and defined the new type of transitive in such a way that it is preserved under topologically $\alpha r$-conjugation.

Proposition 3.5 Let $X$ and $Y$ are $\alpha r$-separable and $\alpha r$-second category spaces. If $f: X \rightarrow X$ and $g: Y \rightarrow Y$ are $\alpha r$-conjugated by the $\alpha r$-homeomorphism $h: Y \rightarrow Y$ then, for each $\alpha r$-hyper-cyclic point $x$ in $X$ it holds that if $h(y)$ is $\alpha r$-hyper-cyclic point in $X$.

Proof: Suppose that $f: X \rightarrow X$ and $g: Y \rightarrow Y$ are maps $\alpha r$-conjugate via $h: Y \rightarrow Y$ such that $h \circ g = f \circ h$, then if $y \in Y$ is $\alpha r$-hyper-cyclic point then $O_f(y) = \{ y, g(y), g^2(y), \ldots \}$ is $\alpha r$-dense in $Y$, let $V \subset X$ be nonempty $\alpha r$-open set. Then since $h$ is a $\alpha r$-homeomorphism, $h^{-1}(V)$ is $\alpha r$-open in $Y$, so there exists $n \in \mathbb{N}$ with $g^n(y) \in h^{-1}(V)$. From $h \circ g^n = f^n \circ h$ it follows that $h(g^n(y)) = f^n(h(y)) \in V$.

So that $O_f(h(y)) = \{ h(y), f(h(y)), f^2(h(y)), \ldots \}$ is $\alpha r$-dense in $X$ so $h(y)$ is $\alpha r$-hyper-cyclic in $X$. Similarly, if $y \in Y$ is $\alpha r$-hyper-cyclic in $Y$. Then

Proposition 3.6 If $f: X \rightarrow X$ and $g: Y \rightarrow Y$ are $\alpha r$-conjugate via $h: X \rightarrow Y$. Then

1. $T$ is $\alpha r$-type transitive subset of $X \Leftrightarrow h(T)$ is $\alpha r$-type transitive subset of $Y$;
2. $T \subset X$ is $\alpha r$-mixing set $\Leftrightarrow h(T)$ is $\alpha r$-mixing subset of $Y$.

Proof (1) Assume that $f: X \rightarrow X$ and $g: Y \rightarrow Y$ are topological systems which are topologically $\alpha r$-conjugated by $h: X \rightarrow Y$. Thus, $h$ is $\alpha r$-homeomorphism (that is, $h$ is bijective and thus invertible and both $h$ and $h^{-1}$ are $\alpha r$-irresolute) and $h \circ f = g \circ h$ Suppose $T$ is $\alpha r$-type transitive subset of $X$. Let $A, B$ be $\alpha r$-open subsets of $Y$ with $B \cap h(T) \neq \emptyset$ and $A \cap h(T) \neq \emptyset$.
(to show $g^n(A) \cap B \neq \emptyset$ for some $n > 0$).

$U = h^{-1}(A)$ and $V = h^{-1}(B)$ are $\alpha$-open subsets of $X$ since $h$ is an $\alpha$-irresolute. Then there exists some $n > 0$ such that $f^n(U) \cap V \neq \emptyset$ since the set $T$ is $\alpha$-type transitive subset of $X$, with $U \cap T \neq \emptyset$ and $V \cap T \neq \emptyset$. Thus

So $h(T)$ is $\alpha$-type transitive subset of $Y$.

Proof (2) We only prove that if $T$ is topologically $\alpha$-mixing subset of $Y$ then $h^{-1}(T)$ is also topologically $\alpha$-mixing subset of $X$. Let $U, V$ be two $\alpha$-open subsets of $X$ with $U \cap h^{-1}(T) \neq \emptyset$ and $V \cap h^{-1}(T) \neq \emptyset$. We have to show that there is $N > 0$ such that for any $n > N$,

$f^n(U) \cap V \neq \emptyset$ and $h^{-1}(V)$ are two $\alpha$-open sets since $h$ is $\alpha$-irresolute with $h^{-1}(V) \cap T \neq \emptyset$ and $h^{-1}(U) \cap T \neq \emptyset$. If the set $T$ is topologically $\alpha$-mixing then there is $N > 0$ such that for any $n > N$, $g^n(h^{-1}(U)) \cap h^{-1}(V) \neq \emptyset$. Then $\exists x \in g^n(h^{-1}(U)) \cap h^{-1}(V)$. That is $x \in g^n(h^{-1}(U))$ and $x \in h^{-1}(V) \Leftrightarrow x = g^n(y)$ for $y \in h^{-1}(U)$, $h(x) \in V$. Thus, since $h \circ g^n = f^n \circ h$, so that $h(x) = h(g^n(y)) = f^n(h(y)) \in f^n(U)$ and we have $h(x) \in V$ that is $f^n(U) \cap V \neq \emptyset$. So, $h^{-1}(T)$ is $\alpha$-mixing set.

Proposition 3.7 Let $(X, f)$ be a topological system and $A$ be a nonempty $\alpha$-closed set of $X$. Then the following conditions are equivalent.

1. $A$ is a $\alpha$-transitive set of $(X, f)$.
2. Let $V$ be a nonempty $\alpha$-open subset of $A$ and $U$ be a nonempty $\alpha$-open subset of $X$ with $U \cap A \neq \emptyset$. Then there exists $n \in \mathbb{N}$ such that $V \cap f^{-n}(U) \neq \emptyset$.
3. Let $U$ be a nonempty $\alpha$-open set of $X$ with $U \cap A \neq \emptyset$. Then $\bigcup_{n \in \mathbb{N}} f^{-n}(U)$ is $\alpha$-dense in $A$.

Theorem 3.8 Let $(X, f)$ be a topological dynamical system and $A$ be a nonempty $\alpha$-closed invariant set of $X$. Then $A$ is $\alpha$-transitive set of $(X, f)$ if and only if $(A, f)$ is $\alpha$-type transitive system.

Proof: $\Rightarrow$) Let $V_1$ and $U_1$ be two nonempty $\alpha$-open subsets of $A$. For a nonempty $\alpha$-open subset $U_1$ of $A$, there exists a $\alpha$-open set $U$ of $X$ such that $U_1 = U \cap A$.

$\phi \neq f^n(h^{-1}(A)) \cap h^{-1}(B) = h^{-1}(g^n(A)) \cap h^{-1}(B)$.

Therefore,

$h^{-1}(g^n(A) \cap B) \neq \phi$ implies $g^n(A) \cap B \neq \emptyset$ since $h^{-1}$ is invertible.

IV. NEW TYPES OF CHAOS IN PRODUCT SPACES

We will give a new definition of chaos for $\delta$-irresoluteself map $f : X \to X$ of a compact Hausdorff topological space $X$, so called $\delta$-type chaos. This new definition induces from John Tylar definition which coincides with Devaney’s definition for chaos when the topological space happens to be a metric space.

Definition 4.1 [4] Let $(X, f)$ be a topological dynamical system; the dynamics is obtained by iterating the map. Then, $f$ is said to be $\delta$-type chaotic on $X$ provided that for any nonempty $\delta$-open sets $U$ and $V$ in $X$, there is a periodic point $p \in X$ such that $U \cap O_f(p) \neq \emptyset$ and $V \cap O_f(p) \neq \emptyset$. 

Proposition 4.2 Let $(X, f)$ be a topological dynamical system. The map $f$ is $\delta$-type chaotic on $X$ if and only if $f$ is $\delta$-type transitive and the periodic points of the map are $\delta$-dense in $X$.

Proof: $\Rightarrow$) If $f$ is $\delta$-type chaotic on $X$, then for every pair of nonempty $\delta$-open sets $U$ and $V$, there is a
periodic orbit intersects them; in particular, the periodic points are $\delta$-dense in $X$. Then there is a periodic point $p$ and $x, y \in O_f(p)$ with $x \in U$ and $y \in V$ and some positive integer $n$ such that $f^n(x) = y$, so that $y = f^n(x) \in f^n(U)$ therefore $f^n(U) \cap V \neq \phi$.

\[ \implies \] The $\delta$-type transitivity [5] of $f$ on $X$ implies, for any nonempty $\delta$-open subsets $U, V \subset X$, there is $n$ such that for some $x \in U$, $f^n(x) \in V$. Now, define $W = f^{-n}(V) \cap U$. Then $W$ is $\delta$-open and nonempty with the property that $f^n(W) \subset V$.

But since the periodic points of $f$ are $\delta$-dense in $X$, there is a $p \in W$ such that $f^n(p) \in V$. Therefore, $U \cap O_f(p) \neq \phi$ and $V \cap O_f(p) \neq \phi$. So, the map $f$ is $\delta$-type chaotic.

We will define some concepts as follows:
1. ($TT_\delta$) if for every non-empty $\delta$-open set $D \subset X$,
   \[ \bigcup_{n=1}^{\infty} f^n(D) \text{ is } \delta \text{-dense}, \]
2. Weak $\delta$-Mixing ($WM_\delta$) if $f \times f$ is topologically $\delta$-transitive.
3. Exact $\delta$-Transitive ($ET_\delta$) if for every pair of non-empty $\delta$-open set $D, W \subset X$,
   \[ \bigcup_{n=1}^{\infty} (f^n(D) \cap f^n(W)) \text{ is } \delta \text{-dense in } X, \]
4. Topologically $\delta$-Mixing ($TM_\delta$) if for every pair of non-empty $\delta$-open set $D, W \subset X$, there exits an $N \in \mathbb{N}$ such that $f^n(D) \cap W \neq \phi$ for all $n \geq N$.
5. $\delta$-Exact ($E_\delta$) if for every non-empty $\delta$-open set $D \subset X$, there exists $N \in \mathbb{N}$ such that $f^N(D) = X$
6. Then the following implications hold:
   - $(E_\alpha) \Rightarrow (ET_\alpha)$;
   - $(TM_\alpha) \Rightarrow (WM_\alpha) \Rightarrow (TT_\alpha)$;

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