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Common Fixed Point Theorems for Self-Maps on Metric Spaces with Weak Distance

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COMMONFIXEDPOINTTHEOREMSFORSELFMAPSONMETRICSPACESWITHWEAKDISTANCE

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Common Fixed Point Theorems for Self-Maps on Metric Spaces with Weak Distance

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I. Introduction

Let $(X;\ d)$ be a metric space. If f is a self-map on X, and $x_0 \in X$, we denote by fx the f-image of x_0.

As a weaker form of the metric d, Kada et al. [5] introduced the notion of weak distance (or simply w-distance) on X as follows:

Definition 1.1. Let $(X;\ d)$ be a metric space and $p : X \times X \to [0, \infty)$ satisfy the following conditions:

(w_1) $p(x, y) \leq p(x, z) + p(z, y)$ for all $x, y, z \in X$

(w_2) For any $x \in X$, $p(x, \cdot) : X \to \mathbb{R}_+$ is lower semi continuous in the second variable, that is $p(x, y_n) \leq \liminf p(x, y_n)$ whenever $y_n \to y$ as $n \to \infty$ for some $x \in X$, and

(w_3) Given $\epsilon > 0$, there is a $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply that $d(x, y) < \epsilon$.

Then p is known as a w-distance on X.

Obviously, every metric d on X satisfies the conditions (w_1)-(w_3), that is d is a w -distance on X.

Example 1.1. Let $X = \left\{\frac{1}{m} : m = 1, 2, 3, \ldots\right\} \bigcup \{0\}$ with metric $d(x,\ y) = x + y$ if $x \neq y$ and $d(x,\ y) = 0$ if $x = y$ for all $x, y \in X$. Note that $(X,\ d)$ is a complete metric space. Define $p(x, y) = y$. Then p a w -distance on X.

For other examples one can refer to [5].

Recently Ume and Sucheol [4] have proved two common fixed point theorems, given below, for self-maps on a complete metric space with a w -distance on X, which generalize and improve the results of Fisher [1], Dien [3] and Liu et al. [6].

Theorem 1.1 ([4], Theorem 3.1). Let X be a complete metric space $(X,\ d)$ with w-distance p on it. Suppose that f, g: $X \to X$ and $\phi : X \to [0, \infty)$ satisfy the conditions:

(a)
$$g(X) \subset f(X) \tag{1.1}$$

(b) there exists $a t \in X$ such that
$$p(t, gx) \leq r p(t, fx) + \phi(fx) - \phi(gx) \quad for\ all \quad x, y \in X, \quad 0 \leq r < 1 \tag{1.2}$$

(c) for every sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X with
$$\lim_{n\to\infty} p(t, fx_n) = 0 = \lim_{n\to\infty} p(t, gx_n), \tag{1.3}$$
we have
$$\lim_{n\to\infty} \max\left\{p(t, fx_n), p(t, gx_n), p(fgx_n, gfx_n)\right\} = 0,$$
and

(d) for each $u \in X$ with $u \neq fu$ or $u \neq gu$
$$\{p(u, fx) + p(u, gx) + p(fgx, gfx) : x \in X\} > 0. \tag{1.4}$$

Then f and g will have a unique common fixed point.

Theorem 1.2 ([4], Theorem 3.6). Let X be a complete metric space $(X,\ d)$ with w-distance p and the mappings f,g: $X \to X$ satisfy the conditions (a) and (d). Suppose that $\phi, \psi : X \to [0, \infty)$ are such that

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(e) for every sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$,

we have $\lim_{n\to\infty} \max\{p(t, fx_n), p(t, gx_n), p(fgx_n, gfx_n)\} = 0$,

and

(f)
$$p(gx, gy) \le a_1 p(fx, fy) + a_2 p(fx, gx) + a_3 p(fy, gy)$$

$$+ a_4 p(fx, gy) + a_5 \sqrt{p(gx, fy) d(fy, gx)} + [\phi(fx) - \phi(gx)] + [\psi(fy) - \psi(gy)] \tag{1.5}$$

for all $x,\ y \in X$ where $a_i \in [0,1),\ i = 1,2,3,4,5$ are such that

$$a_1 + a_4 + a_5 < 1 \text{ and } a_1 + a_2 + a_3 + 2a_4 < 1 \tag{1.6}$$

Then f and g will have a unique common fixed point.

The purpose of this paper is to establish two fixed point theorems, which generalize those of Brian Fisher [1], Dien [3] and Liu et al. [6].

II. Preliminaries

First we state the following lemma, proved in [5]:

Lemma 2.1. Let X be a metric space with w-distance p on it. Then

(g) $p(x,y) = 0$ and $p(x,z) = 0$ imply that $y = z$.

Also $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is *a* Cauchy sequence in X, provided

(h) $p(x_n, x_m) \le \alpha_n$ for all $m > n \ge 1$

(i) $p(x, x_n) \le \alpha_n$ for all $n \ge 1$ for each $x \in X$.

We now introduce an orbit notion that is followed in the rest of the paper.

Definition 2.1. Let f and g be self-maps on X. Given $x_0 \in X$, if there exit points $x_1, x_2, x_3, \dots$ in X such that

$$y_n = gx_{n-1} = fx_n \text{ for } n \ge 1, \tag{2.1}$$

the sequence $\langle y_n \rangle_{n=1}^{\infty}$ is called a g-*orbit* relative to f at x_0 or simply a $(g,\ f)$-orbit at x_0. We call $\langle x_n \rangle_{n=1}^{\infty}$ a *base sequence* associated with the g-*orbit* (2.1). Note that when f is the identity map i on X, (2.1) and the base sequence coincide with the g-*orbit* gx_0, gx_1, ... at x_0. This notion was adopted in [8]. The notion of $(g,\ f)$-orbit is not unique. For instance, Nesic [7] defined a $(g,\ f)$-orbit at x_0 by the iterations:

$$x_{2n-1} = gx_{2n-2},\ x_{2n} = fx_{2n-1} \text{ for } n \ge 1 \tag{2.2}$$

which was employed by Fisher [1] though no name was mentioned.

Remark 2.1. If the self-maps f and g on X satisfy the inclusion (1.1), then by a routine induction, it can be easily shown that $(g,\ f)$-orbit at each x_0 exists with the choice (2.1). Given $x_0 \in X$, there can be more than one base sequence $\langle x_n \rangle_{n=1}^{\infty}$ as the following examples reveal:

Example 2.1. Let $X = \mathbb{R}$ with usual metric $d(x,y) = |x-y|$ for all $x,\ y \in X$. Define $f,\ g: X \to X$ by $f(x) = x_2$ and $g(x) = \frac{x^2}{4}$ for $x \in X$. Then (1.1) is obvious and hence by Remark 2.1, orbits can be specified at each x_0. Given $x_0 \in X$, choose $x_n = \pm\frac{x_0}{2^n}$ for $n \ge 1$. Since each x_n has two choices, several base sequences $\langle x_n \rangle_{n=1}^{\infty}$ can be specified to get the respective (g, f)-orbit.

We now prove

Lemma 2.2. Suppose that $(X,\ d)$ is a metric space with w-distance p on X. Let $f,\ g: X \to X$ and $\phi: X \to [0, \infty)$ satisfy the inclusion (1.1) and the condition (b) of Theorem 1:1. If X is complete metric space and $x_0 \in X$, then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \text{ for some } z \in X. \tag{2.3}$$

Proof. Given $x_0 \in X$, Suppose that $\langle x_n \rangle_{n=1}^{\infty}$ is a base sequence at x_0 and $f,\ g$ are such that (2.1) holds good. Now, by condition (b) with $x = x_{n-1}$, we have

$$p(t, fx_n) = p(t, gx_{n-1}) \le r.p(t, fx_{n-1}) + \phi(fx_{n-1}) - \phi(gx_{n-1})$$

so that for any $k \ge 2$

$$\sum_{n=1}^{k} p(t, fx_n) \le r.\sum_{n=1}^{k} p(t, fx_{n-1}) + \sum_{n=1}^{k} [\phi(fx_{n-1}) - \phi(fx_n)]$$

which gives

$$\sum_{n=2}^{k} p(t, fx_n) \leq \frac{r}{1-r} p(t, fx_1) + \frac{1}{1-r}[\phi(fx_0) - \phi(fx_k)]$$

$$< \frac{r}{1-r} p(t, fx_0) + \frac{1}{1-r}\phi(fx_0)$$

Showing that $\sum_{n=2}^{\infty} p(t, fx_n)$ converges so that n th term tends to 0 as $n \to \infty$, that is $\lim_{n\to\infty} p(t, fx_n) = 0$.

Now, by (i) of Lemma 2.1, it follows that $\langle fx_n\rangle_{n=1}^{\infty}$ is a Cauchy sequence in the $(g,\ f)$-orbit X. Since X is complete, there is a $z \in X$ such that $fx_n \to z$ as $n \to \infty$.

Similar argument shows $\langle gx_n\rangle_{n=1}^{\infty}$ converges to z' in X. Proceeding the limit as $n \to \infty$ in (2.1) and using these limits, it follows that $z = z'$, proving the lemma.

Remark 2.2. The converse of Lemma 2.2 is not true. Infact, the example given below shows that we can find a metric space $(X,\ d)$ with a w-distance p on it satisfying condition (a) and (b) of Theorem 1.1 such that for any $x_0 \in X$ and for any base sequence $\langle x_n\rangle_{n=1}^{\infty}$ at x_0, both $\langle fx_n\rangle_{n=1}^{\infty}$ and $\langle gx_n\rangle_{n=1}^{\infty}$ converge to the same point in X; but X is not complete.

Example 2.2. Let $X = [0,\ 1)$ with $d(x, y) = |x - y|$ for all $x,\ y \in X$. Clearly $(X,\ d)$ is an incomplete metric space. Define $f,\ g : X \to X$ by $fx = \dfrac{2x+1}{4}$ and $gx = \dfrac{1}{2}$ for $x \in X$. Then $g(X) = \left\{\dfrac{1}{2}\right\}$ and $f(X) = \left[\dfrac{1}{4}, \dfrac{3}{4}\right)$ so that $g(X) \subset f(X)$. Let

$$p(x, y) = \frac{1}{4}\max\left\{ |2x-1|, |2x-4y+1|, 2|x-y| \right\} \text{ for } x, y \in X,$$

which will be a w-distance on X. Also for any $t \in X$; $p(t, gx) = \dfrac{1}{4}|2t-1|$ and

$$p(t, fx) = \frac{1}{8}\max\left\{ |2(2t-1)|, |4(t-x)|, |(4t-2x-1)| \right\}$$

from which it follows $p(t, gx) \leq \dfrac{1}{2} p(t, fx) + \phi(fx) - \phi(gx)$ for any $x \in X$ where $\phi(x) = 1$ for all $x \in X$.

Note that for any $x_0 \in X$ there is only one base sequence $\langle x_n\rangle_{n=1}^{\infty}$ given by $x_n = \dfrac{1}{2}$ for all $n \geq 1$ so that both $\langle fx_n\rangle_{n=1}^{\infty}$ and $\langle gx_n\rangle_{n=1}^{\infty}$ are constant sequences with each term equal to $\dfrac{1}{2}$; and hence they converge to $\dfrac{1}{2} \in X$.

Lemma 2.3. Suppose $(X,\ d)$ is a metric space with w-distance p on it. Let $f,\ g : X \to X$ and $\phi, \psi : X \to [0,\infty)$ be such that (a) of Theorem 1.1 and (f) of Theorem 1.2 hold. If $(X,\ d)$ is a complete metric space, then for any $x_0 \in X$ and for any base sequence $\langle x_n\rangle_{n=1}^{\infty}$ at x_0, both the sequences $\langle fx_n\rangle_{n=1}^{\infty}$ and $\langle gx_n\rangle_{n=1}^{\infty}$ converge to the same point in X.

Proof. Suppose that $\langle x_n\rangle_{n=1}^{\infty}$ is a base sequence at some $x_0 \in X$ with the choice (2.1). Write

$$\gamma_n = p(fx_n, fx_{n+1}) = p(gx_{n-1}, gx_n \text{ for } n \geq 1$$

Then by (f) of Theorem 1.2, we have

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$$
\begin{aligned}
\gamma_n &= p(gx_{n-1}, gx_n)\\
&\leq a_1 p(fx_{n-1}, fx_n) + a_2 p(fx_{n-1}, gx_{n-1}) + a_3 p(fx_n, gx_n)\\
&\quad + a_4 p(fx_{n-1}, gx_n) + a_5\sqrt{p(gx_{n-1}, fx_n) d(fx_n, gx_{n-1})}\\
&\quad + [\phi(fx_{n-1}) - \phi(gx_{n-1})] + [\ \ (fx_n) - \ \ (gx_n)]\\
&\leq a_1\gamma_{n-1} + a_2\gamma_{n-1} + a_3\gamma_n + a_4(\gamma_{n-1} + \gamma_n)\\
&\quad + [\phi(fx_{n-1}) - \phi(fx_n)] + [\ \ (fx_n) - \ \ (fx_{n+1})]\\
&= (a_1 + a_2 + a_4)\gamma_{n-1} + (a_3 + a_4)\gamma_n\\
&\quad + [\phi(fx_{n-1}) - \phi(fx_n)] + [\ \ (fx_n) - \ \ (fx_{n+1})]
\end{aligned}
$$

from which we get

$$\gamma_n \quad \leq \alpha\gamma_{n-1} + \beta\{[\phi(fx_{n-1}) - \phi(fx_n)] + [\ \ (fx_n) - \ \ (fx_{n+1})]\},\ n \geq 2,$$

where $\alpha = \dfrac{a_1 + a_2 + a_4}{1 - a_3 - a_4}$ and $\beta = \dfrac{1}{1 - a_3 - a_4}$

Therefore for any integer $k \geq 2$,

$$\sum_{n=2}^{k}\gamma_n \leq \alpha\sum_{n=2}^{k}\gamma_{n-1} + \beta\Big[\phi(fx_1) - \phi(fx_k)] + [\ \ (fx_2) - \ \ (fx_3)\Big]$$

which gives

$$\sum_{n=2}^{k}\gamma_n \leq \frac{\gamma_1\alpha}{1-\alpha} + \frac{\beta\big[\phi(fx_1) + \ \ (fx_2)\big]}{1-\alpha}$$

showing that $\sum_{n=2}^{\infty}\gamma_n$ converges and hence $_n \to 0$ as $n \to \infty$. Also from the above lines, for $m > n \geq 2$, we see that $p(fx_n, fx_m) \leq \alpha_n$ where $\alpha_n = \gamma_n + \gamma_{n+1} + \cdots + \gamma_{m-1}$. Since $\alpha \to 0$ as $n \to \infty$' it follows from (h) of Lemma 2.1 that $\langle fx_n\rangle_{n=1}^{\infty}$ is a Cauchy sequence in X and hence converges to some $z \in X$. Similarly we can prove that $\langle gx_n\rangle_{n=1}^{\infty}$ converges to some z' in X. But $gx_{n-1} = fx_n$ for all $n \geq 1$ it follows that $z = z'$, completing the proof of lemma.

Remark 2.3. The converse of Lemma 2.3 is not true. In fact, it is not difficult to exhibit a metric space (x, d) with w-distance p on it for which (a) of Theorem 1.1 and (f) of Theorem 1.2 in which for any $x_0 \in X$ and for any base sequence $\langle x_n\rangle_{n=1}^{\infty}$ at x_0 both sequences $\langle fx_n\rangle_{n=1}^{\infty}$ and $\langle gx_n\rangle_{n=1}^{\infty}$ converge to the same point, yet (x; d) is not complete.

III. Main Results

Theorem 3.1. let (x, d) be a metric space with w-distance p on it. Suppose that $f, g : X \to X$ and $\phi : X \to [0, \infty)$ satisfy the inclusion (1.1) and the condition (1.4) of Theorem 1:1. Also suppose that

(j) there is a base sequence $\langle x_n\rangle_{n=1}^{\infty}$ at some point $x_0 \in X$ such that $\langle fx_n\rangle_{n=1}^{\infty}$ and $\langle gx_n\rangle_{n=1}^{\infty}$ converge the same point $z \in X$

(k) $p(z, gx) \leq rp(z, fx) + \phi(fx) - \phi(gx)$ for all $x \in X$, where $0 \leq r < 1$ and

(l) for every sequence $\langle u_n\rangle_{n=1}^{\infty} \subset X$ with $\lim_{n\to\infty} p(z, fu_n) = \lim_{n\to\infty} p(z, gu_n) = 0$, we have

$$\lim_{n\to\infty} \max\{p(z, fu_n), p(z, gu_n), p(fgu_n, gfu_n)\} = 0. \tag{3.1}$$

Then z is a unique common fixed point of f and g.

Proof. Writing $x = x_n$ in (k) we get

$$p(z, fx_{n+1}) = p(z, gx_n) \leq rp(z, fx_n) + \phi(fx_n) - \phi(gx_n).$$

Then as in Lemma 2.2, we can prove that $\sum_{n=1}^{\infty} p(z, fx_n)$ converges hence

$$\lim_{n\to\infty} p(z, fx_n) = \lim_{n\to\infty} p(z, gx_n) = 0.$$

Using this in (3.1), it follows that $\lim_{n\to\infty} p(fgx_n, gfx_n) = 0.$.

Now if z is not a common fixed point of f and g, then either $fz \neq z$ or $gz \neq z$; and therefore, by the condition (1.4) of Theorem 1:1

$$\begin{aligned} &0 < \inf\{p(z, fx) + p(z, gx) + p(fgx, gfx) : x \in X\} \\ &\leq \inf\{p(z, fx_n) + p(z, gx_n) + p(fgx_n, gfx_n) : n \geq 1\} \\ &= 0, \end{aligned}$$

a contradiction. Hence $fz \neq z$ and $gz \neq z$.

The uniqueness of the common fixed point z follows as in the proof of Theorem 3.1 of [4].

Remark 3.1. In view of Remark 2.3, Theorem 3.1 generalizes Theorem 1.2. Also since d is a w-distance, the results proved by Dien [3] and Liuet.al [6] will be particular cases of Theorem 3.1.

Theorem 3.2. Let (X, d) be a metric space with w-distance p on it. Suppose that $f, g : X \to X$ and $\phi : X \to [0,\infty)$ satisfy the inclusion (1.1) and the condition (1.4) of Theorem 1:1 and the condition (*f*) of Theorem 1.2. If (*j*) and *l*) hold good, then z is the unique common fixed point of f and g.

Proof. The proof is similar to the first main result and is omitted here.

Remark 3.2. In view of Remark 2.6, Theorem 3.2 generilzes Theorem 1.2. Also since d is a w-distance on X, the fixed point theorem of Fisher [1] is a partcular case of Theorem 3.2 with $p = d$.

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