Analytical Solution of a Guided Simply Supported Beam under Three Point Bending

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Abstract

In the present paper, elastic field parameters of a guided simply supported beam under three point bending has been investigated by a new approach called displacement potential formulation. Guided beams involve mixed mode of boundary conditions which the classical beam theory cannot handle. In displacement potential formulation, all the elastic field parameters has been expressed in terms of a single function ? which gives rise to a fourth order partial differential equation. The solution of this equation has been determined by trial and error process satisfying all the boundary conditions. This solution is then inserted to reduce the partial differential equation into an ordinary one. The obtained analytical solution of the beam has been presented as graphs. The stress concentration occurs mainly at the point of application of the load and the support at two corners. The analytical solution has further been verified by finite element analysis.

Index terms— elastic field parameters, displacement potential formulation, mixed mode of boundary conditions, stress concentration, finite element analysis.

1 Introduction

beam may be considered as one of the most commonly used structural elements in engineering applications. A beam is said to be a deep beam when the depth is comparable to its span. Design of deep beams based on classical Euler bending theory can be seriously erroneous, since the simple theory of flexure takes no account of the effect of normal pressures on the top and bottom edges of the beam caused by the loads and reactions (Chow, Conway and Winter, 1952). The effect of normal pressures on the stress distribution in deep beams is such that the distribution of bending stresses on vertical sections is not linear and the distribution of shear stresses is not parabolic. Consequently, a plane transverse section does not remain plane after bending, and the neutral axis does not lie at the mid-depth, which eventually causes the basis of classical theory to be violated.

In an attempt to make up the limitation, different theories as well as methods of solution have been reported in the literature (Conway, Chow and Morgan, 1951;Conway and Ithaca, 1953;Murty, 1984;Suzuki, 1986). However, each solution possesses certain limitations, and eventually none of the solutions are found to conform to all the physical characteristics of the problem for deep beam appropriately. Even, photoelastic studies (Uddin, 1966), finite element analysis (Hardy and Pipelzadeh, 1991) and finite difference solutions (Ahmed, Idris and Uddin, 1966; Ahmed, Khan and Uddin, 1998; Ahmed, Idris and Uddin, 1999; Ahmed, Hossain and Uddin 2005; Akanda, Ahmed and Uddin, 2002) have also been carried out for deep beams on two supports, mainly because all the physical conditions imposed on the beam could not be fully taken into account in the analytical methods of solution. Among the existing mathematical models of elasticity for the plane boundary-value problems, the stress function approach and the displacement formulation are noticeable. The stress function approach accepts boundary conditions in terms of loading only; boundary restraints cannot be satisfactorily imposed on it. On the other hand, the displacement formulation involves extreme difficulty especially when the boundary conditions are a mixture of restraints and stresses. As a consequence, serious attempts had hardly been made in the past for stress analysis using this formulation. As such, neither of the existing formulations is suitable for solving problems of mixed boundary conditions.

Further, the use of standard structures, like beams, columns, etc. with guides on part or full of their bounding surfaces is receiving increased importance in order to satisfy precise and strict design criteria in many of the engineering applications. Guided boundaries usually help in reducing the level of deformation in the structural elements, which eventually resist the change of the original shape of the bounding surfaces under loading. But structures with guided boundaries always remain away from the scope of analytical solutions, because the physical conditions of guided boundaries need to be mathematically modelled in terms of a mixed mode of boundary conditions.

Since the exact analytical solution of mixedboundary-value elastic problems, is beyond the scope of existing mathematical models of elasticity, the use of a new mathematical formulation will be investigated to analyze the elastic behavior of a guided deep beam under three point bending loading and support arrangements. It would be worth mentioning that, as far as the reporting in the literature is concerned, the author has not come across any reliable study of the present problem. Therefore, the analytical solution for a guided deep beam under three point bending is yet to be developed.

2 II.

3 Boundary Conditions

The physical conditions at different boundaries of the beam are expressed mathematically as follows:

? u x = 0 at the edge of x = 0 ? u x = 0 at the edge of x = L ? ? xy (0,y) = 0 at the edge of x = 0 ? ? xy (L,y) = 0 at the edge of x = L ? ? xy (0,y) = 0 at the edge of y = 0 ? ? xy (0,y) = 0 at the edge of y = D ? The lateral stress at the edge of y = D is related to the applied load for the three point bending. Since the point load is actually acting over a certain area of the beam, for instance it can be considered for the length of x=0.45L to 0.55L. Again it is considered that the load intensity is ? o . Therefore, the magnitude of point load, P= 0.1L? o . Then for x=0.45L to 0.55L

? Similarly, the lateral stress at the edge of y = 0 is related to the reactions at the support. In this case ? yy (x,0) = ? o /2 for x=0 to 0.1L and 0.9L to L.

4 Analytical Solution

The equation of equilibrium for isotropic material is as follows (Timoshenko and Goodier, 1970):0 2 4 4 2 2 4 4 4 = ? ? + ? ? ? + ? ? y y x x ? ? ? (1)

The expressions of displacement and stress components in terms of function ?(x, y) are as follows (Nath, Ahmed and Afsar, 2006):

( )y x y x u x ? ? ? = ? 2 , (2a) ( ) ( ) ? ? ? ? ? ? ? ? ? ? + ? ? + ? = 2 2 2 2 1 2 1 1 , y x y x u y ? µ ? µ (2b) ( ) ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? + ? = 2 2 2 3 2 1 , y y x E y x xx ? µ ? µ ? (2c) ( ) ( ) ( ) ? ? ? ? ? ? ? ? + ? ? ? + + ? = 3 3 2 3 2 2 1 , y y x E y x yy ? ? µ µ ? (2d) ( ) ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? + ? = 2 3 3 3 2 1 , y x x E y x xy ? µ ? µ ? (2e)

The potential function ?(x, y) is first assumed in a way so that the physical conditions of the two opposing guided ends are automatically satisfied. At the same time solution has to satisfy the 4 th order partial differential equation. After a long trial and error process, the solution of the governing equation (**??**) is thus approximated as follows:cos) ( ) , (y K x y Y y x m m + = ? ? = ? ?**(3)**

where,) (y f Y m = ,) L / m (? = ?

, K is an arbitrary constant and m= 1, 2, 3, ??.. ? . 3) with respect to x and y are substituted in Eq. (**??**) and following equation is obtained:

5 Derivatives of equation (

Y m //// ? 2? 2 Y m // + ? 4 Y m = 0 (4)

The solution of the above 4 th order ordinary differential equation with constant coefficients [Eq. (**??**)] can normally be approximated as follows: ( ) Now, the reactions on the bottom boundary (y = 0) are acting over the two supports. It is considered that the supports are located at x=0 to 0.1L and x=0.9L to L respectively. The total length for reaction is 20 percent of beam length. Now the compressive load exerted at the mid-span on the edge D y = of the beam may be considered as acting over at least some length of the beam, for instance x= 0.45L to 0.55L. As a result the intensity of reaction is half of the load intensity. Therefore, the reactions over the beam at the supports can be taken as Fourier series in the following manner: y x y x u x ? ? ? = ? 2 , ( ) ( ) [ ] x e y D e C e y B e A m y m y m y m y m ? ? ? ? ? ? ? ? ? ? sin 1 1 1 ? ? = ? ? ? ? ? + + ? = (7a) ( ) y x u y , ( ) ( ) ? ? ? ? ? ? ? ? ? ? + ? ? + ? = 2 2 2 2 1 2 1 1 y x ? µ ? µ ( ) ( ) ( ) ( ) ( ) ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? + ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? + ? ? + + ? + ? ? ? + + ? + ? = ? ? = ? ? 1 2 2 1 6 cos 2 2( ) ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? + = 3 3 2 3 2 1 , y y x E y x xx ? µ ? µ ? ( ) ( ) ( ) ( ) ( ) ? ? ? ? ? ? ? ? ? + ? ? ? ? ? ? ? ? ? ? + + ? ? + + ? + + + + + + ? = ? ? = ? ? K x e y y D e C e y y B e A E m y m y m y m y m µ ? ? µ µ? ? µ ? µ µ? ? µ ? µ ? ? ? ? 6 cos 1 3 1 1 3 1 1 2 1 2 (7c) ( ) ( ) ( ) ? ? ? ? ? ? ? ? ? + ? ? ? + + ? = 3 3 2 3 2 2 1 , y y x E y x yy ? ? µ µ ? ( ) ( ) ( ) ( ) ( ) ? ? ? ? ? ? ? ? ? + ? ? ? ? ? ? ? ? ? ? ? + ? + +

+ + + ? ? ? + ? ? + ? = ? ? = ? ? K x e y y D e C e y y B e A E m y m y m y m y m 6 cos 1 1 1 1 2 1 2 ? ? µ µ? ? µ ? µ µ? ? µ ? µ ? ? ? ? (7d) ( ) ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? + ? = 2 3 3 3 2 1 , y x x E y x xy ? µ ? µ ? ( ) ( ) ( ) ( ) ( ) ? + + + + + + + + + + ? = ? ? = ? ? x e? = 20 9 sin 20 11 sin 2 0 L L L ? = 20 9 sin 20 11 sin 2 0 ? ? ? ? m m m (8b)

The reaction load at the support on the edge 0 = y can also be given by a Fourier series as follows: and using Eq. (9a) and Eq. (12) the arbitrary constant K can be obtained as follows:L L L m xdx xdx L E ? ? ? ? ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? ? + ? ? ? ? ? ? ? ? = 10 9 sin sin 0 10 sin 0 L L L L ? ? ? ? ? ? ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? + ? ? ? ? ? ? ? =**10**0 = y ( ) ( ) ( ) ( ) { } ( ) ( ) ? ? ? ? = ? = ? = + = ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? + ? ? ? ? ? ? ? ? ? ? + ? + + + + ? + + + ? + ? 1 1 2 3 2 3 2 1 2 cos 6 cos 3 3 cos 2 1 m o m m m m m m m m m m m E x E K x D C B A x D C B A E ? ? ? ? ? ? ? ? ? ? µ µ**(12) Therefore, ( ) ( ) ( ) ( )**( ) [ ] m m m m m E D C B A E ? + ? + + + µ ? µ µ ? µ µ ? 1 1 1 1 1

( )10 6 1 0 0 2 ? µ = = + ? E K E or, ( ) E K 60 1 2 0 µ ? + ? = (15)

The simultaneous equations (10), (**??**1), (**??**3) and (**??**4

6 (

)2 2 11 1 µ ? + ? = E Z

Therefore, stress and displacement components at various points of the beam can be obtained using equations of (7).

IV.

7 Result Analysis

The analytical solutions of displacement and stress components are obtained for various aspect ratios (L/D) of the beam. The material of the beam is mild steel whose modulus of elasticity is E=209 X 10^9 and poison's ratio ?=0.3. The result of a guided isotropic beam having aspect ratio two and the uniform loading parameter 0 ? = 40 N/mm is presented in sequence of axial displacement (u x), lateral displacement (u y), bending stress (? xx), normal stress (? yy) and shearing stress (? xy). Axial displacements (u x) are found to be zero at the mid-section of span and at the lateral guided boundaries. Zero value of u x at the guided ends verifies the boundary condition of those edges of the beam. Axial displacements distribution is found skewsymmetric about the mid-span of the beam. The values of u x for sections 0<x/L< 0.5 are negative at the lower portion and positive at upper portion of the beam. The maximum magnitudes of u x /L= ±0.000158 are observed on bottom fiber at the sections of x/L = 0.1 and x/L = 0.9 respectively.

Lateral displacements (u y) are found to take positive value near the two guided lateral ends and negative in the region 0.25<x/L< 0.75 for L/D=2 The u y results are in confirmation to the physical condition of the beam. The beam is being pushed up at the corners and forced down at the mid-span region. The normalized values of positive and negative maximum lateral displacements are u y /D= 0.000250 and u y /D= -0.000426 respectively for L/D=2. The maximum magnitude is observed on the topmost fiber at the midspan. All four edges and mid-span section of the guided simply supported beam are found free from shearing stress. The distribution of shearing stress (? xy) for point loading is anti-symmetric in two sides about the mid-span of the beam. The maximum concentration of shearing stress is observed near the bottom corners at the supports and is at the top edge where the termination of loading takes place. The normalized maximum magnitude of shear stress is ±0.2 for L/D= 2. Shear stress distribution at transverse section is nearly parabolic. Along beam depth maximum shear occurs at x/L=0.2 and normalized value of this shear stress is -0.18.Shear stress is zero at x/L=0.4.

As the aspect ratio increases, the magnitude of both axial and lateral displacement increases. The sharp changes in the curve become gradually smoother for higher aspect ratios.

8 Verification

Finite element analysis has also been carried out to verify the stress component of the beam. It can be observed that the stresses found out by the analytical solution, is in complete harmony to that of the finite element analysis. Hence the validity of the displacement potential formulation is justified.

9 Conclusion

Analytical solution using displacement potential approach for the elastic fields of a guided simply supported beam of isotropic material under three point bending is explored satisfying all the physical conditions of the beam appropriately. The specialty of the guided ends is the mixed mode of boundary conditions.

Basically, the guided ends provide the freedom of lateral displacement but not the axial one. At this scenario the necessity of imposing boundary restraints is essential. But it is not practicable to use the classical Bernoulli-Euler beam theory for the solution of guided beam. Because, it cannot handle mixed mode of boundary conditions. Displacement potential formulation can handle mixed mode of boundary condition appropriately and we have found the solution of the beam. It is observed from the solution of the beam that a) Axial displacement is maximum at bottom fiber i.e. y/D=0.0.As the aspect ratio increases axial displacement also increases.

b) Lateral displacement is maximum at top fiber i.e. y/D=1.0. As the aspect ratio increases lateral displacement also increases. So if failure occurs it will occur at midsection of the beam i.e. where deflection is maximum. c) Maximum stress concentration occurs at midsection of the beam .So in terms of stress, midsection is more vulnerable to failure for a simply supported beam. These findings will have applications in aircraft, spacecraft and vehicle structures for predicting appropriate stress distribution in them, thus allowing designers to design with greater safety. [1]

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Figure 1: Fig. 1 :Fig. 2 :

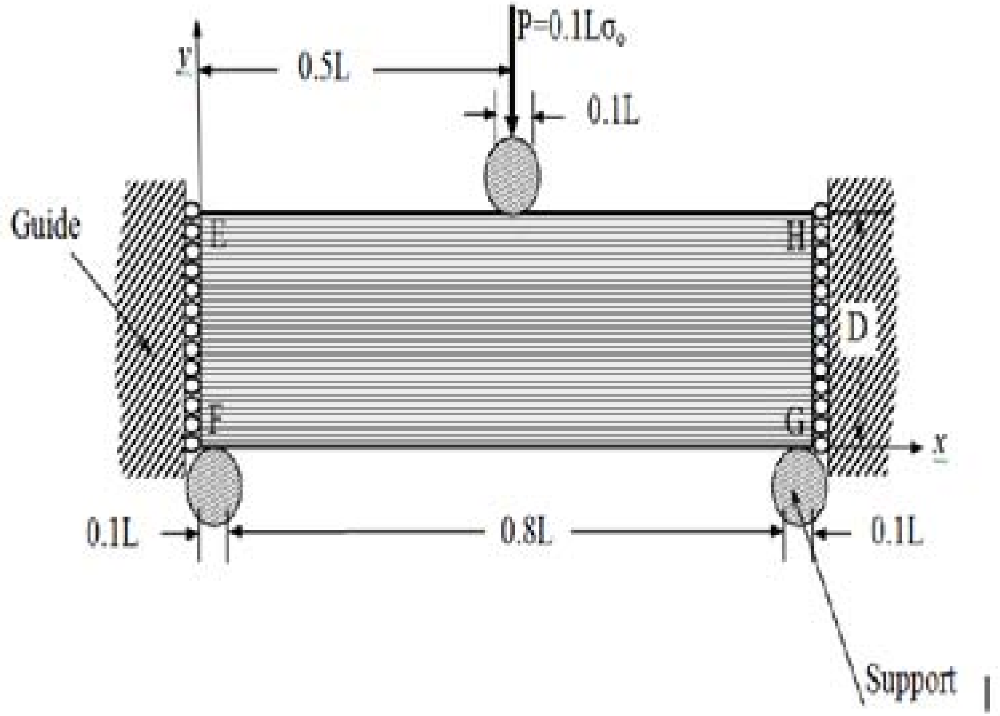

Figure 2:

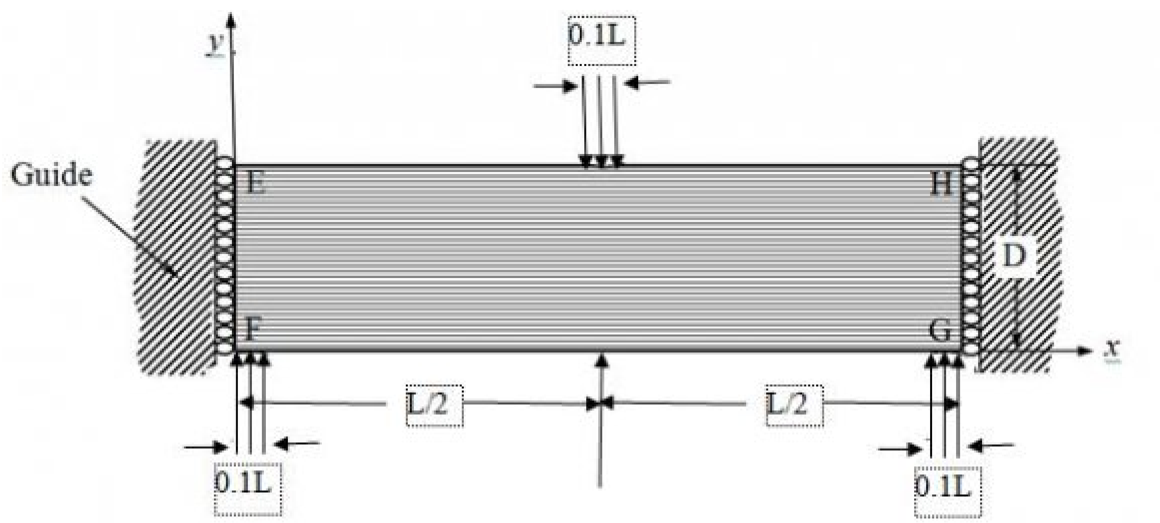

Figure 3:

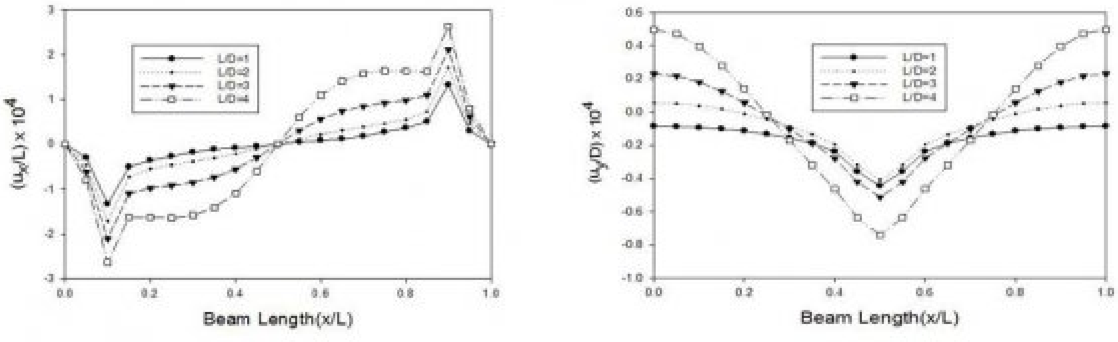

Figure 4:

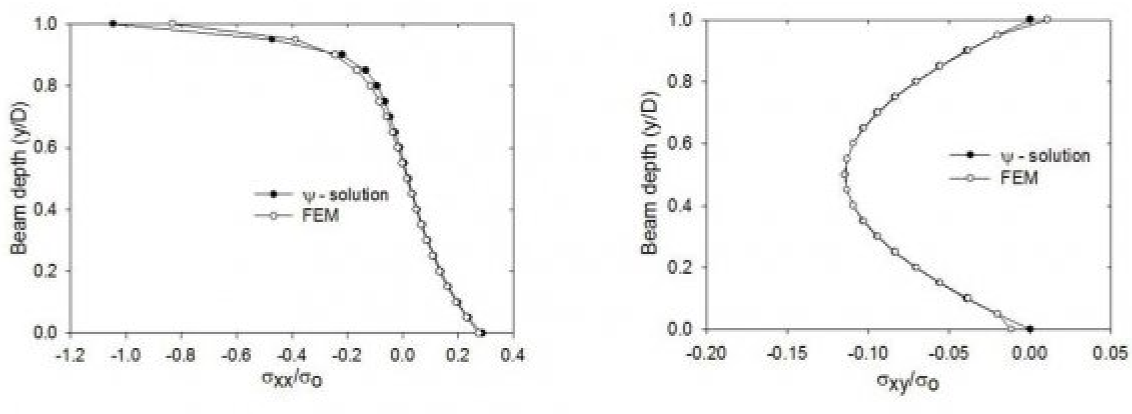

Figure 5:

[Ahmed et al. ()] 'A general mathematical formulation for finite-difference solution of mixed-boundary-value problems of anisotropic materials'. S R Ahmed , M Z Hossain , M W Uddin . *Computers & Structures* 2005. 83 p. .

[Ahmed et al. ()] 'An alternative method for numerical Solution of mixed boundary value elastic problems'. S R Ahmed , A B M Idris , M W Uddin . *Journal of Wave-Material Interaction* 1999. 14 p. .

[Conway et al. ()] 'Analysis of deep beams'. H D Conway , L Chow , G W Morgan . *Journal of Applied Mechanics* 1951. 18 (2) p. .

[Chow et al. ()] L Chow , H D Conway , G Winter . *Stresses in Deep Beam. Trans ASCE*, 1952. (Paper 2557)

[Nath et al. ()] *Displacement potential solution of short stiffened flat composite bars under axial loadings*, S K D Nath , S R Ahmed , A M Afsar . 2006.

[Uddin ()] *Finite Difference Solution of Two-dimensional Elastic Problems with mixed Boundary Conditions*, M W Uddin . 1966. Carleton University, Canada (M. Sc. Thesis)

[Ahmed et al. ()] 'Investigation of stresses at the fixed end of deep cantilever beams'. S R Ahmed , M R Khan , K M S Islam , M W Uddin . *Computers & Structures* 1998. 69 p. .

[Ahmed et al. ()] 'Numerical solution of both ends fixed deep beams'. S R Ahmed , A B M Idris , M W Uddin . *Computers & Structures* 1996. 61 (1) p. .

[Conway and Ithaca ()] 'Some problems of orthotropic plane stress'. H D Conway , N Y Ithaca . *Journal of Applied Mechanics* 1953. 52 p. .

[Hardy and Pipelzadeh ()] 'Static analysis of short beams'. S J Hardy , M K Pipelzadeh . *Journal of strain Analysis* 1991. 26 (1) p. .

[Akanda et al. ()] 'Stress analysis of gear teeth using displacement potential function and finite differences'. M A S Akanda , S R Ahmed , M W Uddin . *International Journal for Numerical Methods in Engineering* 2002. 53 p. .

[Suzuki ()] 'Stress analysis of short beams'. S Suzuki . *AIAA Journal* 1986. 24 p. .

[Timoshenko and Goodier ()] S P Timoshenko , J N Goodier . *Theory of Elasticity*, (New York) 1970. McGraw Hill. (3rd ed.)

[Murty ()] 'Towards a consistent beam theory'. A V K Murty . *AIAA Journal* 1984. 22 p. .