

An Exponential Time Differencing Method for the Kuramoto-Sivashinsky Equation

Gentian Zavalani¹

¹ Polytechnic University of Tirana

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Abstract

The spectral methods offer very high spatial resolution for a wide range of nonlinear wave equations, so, for the best computational efficiency, it should be desirable to use also high order methods in time but without very strict restrictions on the step size by reason of numerical stability. In this paper we study the exponential time differencing fourthorder Runge-Kutta (ETDRK4) method; this scheme was derived by Cox and Matthews in [S.M. Cox, P.C. Matthews, Exponential time differencing for stiff systems, J. Comp. Phys. 176 (2002) 430-455] and was modified by Kassam and Trefethen in [A. Kassam, L.N. Trefethen, Fourth-order time stepping for stiff PDEs, SIAM J. Sci. Comp. 26 (2005) 1214-1233]. We compute its amplification factor and plot its stability region, which gives us an explanation of its good behavior for dissipative and dispersive problems. We apply this method to the Kuramoto-Sivashinsky Equation obtaining excellent results.

Index terms— kuramoto-sivashinsky, etd, ETDRK4, stiff systems, integrating factor.

1 Introduction

The spectral methods have been shown to be remarkably successful when solving time-dependent partial differential equations (PDEs). The idea is to approximate a solution $u(x, t)$ by a finite sum $\sum_{k=0}^N \hat{u}_k e^{ikx} e^{-\lambda_k t}$ where the function class \hat{u}_k , $0, 1, 2, \dots, k, N =$

will be trigonometric for x -periodic problems and, otherwise, an orthogonal polynomial of Jacobi type, with Chebyshev polynomial being the most important special case. To determine the expansion coefficients \hat{u}_k , we will focus on the pseudospectral methods, where it is required that the coefficients make the residual equal zero at as many (suitably chosen) spatial points as possible. Three books [15, 17] and [19] have been contributed to supplement the classic references [18] and [16]. When a time-dependent PDE is discretized in space with a spectral simulation, the result is a coupled system of ordinary differential equations (ODEs) in time: it is the notion of the method of lines and the resulting set of ODEs is stiff; the stiffness problem may be even exacerbated sometimes, for example, using Chebyshev polynomials. The linear terms are primarily responsible for the stiffness with rapid exponential decay of some modes (as with a dissipative PDE) or a rapid oscillation of some modes (as with a dispersive PDE). Therefore, for a time-dependent PDE which combines low-order nonlinear terms with higher-order linear terms it is desirable to use higher-order approximation in space and time. The outline of this paper is as follows. In Section 2 we describe the ETDRK4 (Exponential Time Differencing fourth-order Runge-Kutta) method by Cox and Matthews in [12] and the modification proposed by Kassam and Trefethen in [6]. We discuss the stability of the ETDRK4 method in Section 3. In Sections 4 and 5 we test the method for the Kuramoto-Sivashinsky equation in one space dimensions and, finally, In Sections 6 we summarize our conclusions.

2 II.

3 Exponential Time Differencing Fourth-Order Runge-Kutta Method

The numerical method considered in this paper is an exponential time differencing (ETD) scheme. These methods arose originally in the field of computational electrodynamics [20]. Since then, they have recently received attention in [21] and [22], but the most comprehensive treatment, and in particular the ETD with Runge-Kutta time stepping, is in the paper by Cox and Matthews [12]. The idea of the ETD methods is similar to the method of the integrating factor (see, for example, [15] or [19]) we multiply both sides of a differential equation by some integrating factor, then we make a change of variable that allows us to solve the linear part exactly and, finally, we use a numerical method of our choice to solve the transformed nonlinear part.

When a time-dependent PDE in the form $(\cdot, \cdot)_{t u u u t} = + ? L$ (2.1) where $?$ and $?$ are the linear and nonlinear operators respectively, is discretized in space with a spectral method, the result is a coupled system of ordinary differential equations (ODEs), (\cdot, \cdot) Where the linear term is gone; now we can use a time stepping method of our choice to advance in time. However, the integrating factor methods can also be a trap, for example, to model the formation and dynamics of solitary waves of the KdV equation (see Chapter 14 of [15]). A second drawback is the large error constant. In the derivation of the ETD methods, following [21], instead of changing the variable, we integrate (2.3) over a single time step of length, getting The various ETD methods come from how one approximates the integral in this expression. Cox and Matthews derived in [12] a set of ETD methods based on the Runge-Kutta time stepping, which they called ETDRK methods. In this paper we consider the ETDRK4 fourth-order scheme with the formulae $1/2 (1 - \tau L) (1 + \tau L) Lh Lh n n n n a e u L e I N u t ? = + ? / 2 (1 - \tau L) (1 + \tau L) Lh Lh n n n n h b e u L e I N a u t ? = + ? + / 2 (1 - \tau L) (1 + \tau L) (1 + \tau L) Lh Lh n n n n n h b e a L e I N b t N u t ? ? ? = + ? + ? ? ? ? ? \{ 2 3 2 1 2 4 (4 3 ()) (,) 2 [2 (2)] (,) (,) 2 2 4 3 ? ? = + ? ? ? ? ? + ? ? ? ? ? + ? ? ? ? ? + ? ? ? ? ? + ? ? ? ? ? \}$

More detailed derivations of the ETD schemes can be found in [12]. Unfortunately, in this form ETDRK4 suffers from numerical instability when L has eigenvalues close to zero, because disastrous cancellation errors arise. Kassam and Trefethen have studied in [6] these instabilities and have found that they can be removed by evaluating a certain integral on a contour that is separated from zero. The procedure is basically to change the evaluation of the coefficients, which is mathematically equivalent to the original ETDRK4 scheme of [12], but in [23] it has been shown to have the effect of improving the stability of integration in time. Also, it can be easily implemented and the impact on the total computing time is small. In fact, we have always used this idea in our MATLAB © codes.

III.

4 On the Stability of Etdrk4 Method

The stability analysis of the ETDRK4 method is as follows (see [21,24] or [12]). For the nonlinear ODE $(\cdot, \cdot) (t) (t t u F t c u d t t d u + = (3.1)$ With (\cdot, \cdot)

For the nonlinear part, we suppose that there exists a fixed point $0 u$ this means that $0 (0, 0) c u F u t + = .$ Linearizing about this fixed point, if $(\cdot) u t$ is the perturbation of $0 u$ and $0 '(,) F u t ? =$ then $(\cdot) () () d u t c u t u t d t ? = + (3.2)$

and the fixed point $0 () u t$ is stable if $Re() 0 c ? + < .$

The application of the ETDRK4 method to (3.1) by the above expressions suffers from numerical instability for y close to zero. Because of that, for small y , instead of them, we will use their asymptotic expansions. We make two observations:

As $0 y ? y$, our approximation becomes $2 3 4 1 1 1 () 1 2 6 2 4 r x x x x x = + + + +$ which is the stability function for all the 4-stage Runge-Kutta methods of order four.

Because c and $?$ may be complex, the stability region of the ETDRK4 method is four-dimensional and therefore quite difficult to represent. Unfortunately, we do not know any expression for $(,) 1 r x y =$ we will only be able to plot it. The most common idea is to study it for each particular case; for example, assuming c to be fixed and real in [21] or that both c and $?$ are pure imaginary numbers in [24]. $1 () 2 t x x x u u u ? = ? [,] x ? ? ? ?$

where $0 1 ? < ? (3.4)$ Transforming it to the Fourier space gives In fact, for $0 y < , 1 y \ll$ the boundaries that are observed approach to ellipses whose parameters have been fitted numerically with the following result. $x = ,$ the intersection with the imaginary axis $Im() x$ increases as y , i.e., as $2 ? .$ Since the boundary of stability grows faster than $,$ the ETDRK4 method should have a very good behavior to solve Burger's equation, which confirms the results of paper [6].

IV.

5 The Kuramoto-Sivashinsky Equation

The Kuramoto-Sivashinsky equation (K-S), is one of the simplest PDEs capable of describing complex behavior in both time and space. This equation has been of mathematical interest because of its rich dynamical properties.

100 In physical terms, this equation describes reaction diffusion problems, and the dynamics of viscous-fluid films
101 flowing along walls.

102 Kuramoto-Sivashinsky equation in one space dimension can be written $u_t + u u_x = \nu u_{xx} - u^3$ (4.2)

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104 The Kuramoto-Sivashinsky equation with $2L$ periodic boundary conditions in Fourier space can be written as
105 follows $u_t + u u_x = \nu u_{xx} - u^3$
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109 By simplifying and note that $u_t + u u_x = \nu u_{xx} - u^3$
110 $u_t + u u_x = \nu u_{xx} - u^3$

111 Equation (4.1) can be written as follows $u_t + u u_x = \nu u_{xx} - u^3$ (4.9) Where $u_t + u u_x = \nu u_{xx} - u^3$
112 $u_t + u u_x = \nu u_{xx} - u^3$

113 In final form will be $u_t + u u_x = \nu u_{xx} - u^3$ (4.9)

114 Equation has strong dissipative dynamics, which arise from the fourth order dissipation The stiffness in the
115 system (4.9) is due to the fact that the diagonal linear operator A

116 ,with the elements A_{kk} , has some large negative real eigenvalues that represent decay, because of the strong
117 dissipation, on a time scale much shorter than that typical of the nonlinear term. The nature of the solutions
118 to the the Kuramoto-Sivashinsky equation varies with the system size of linear operator. For large size of
119 linear operator, enough unstable Fourier modes exist to make the system chaotic. For small size of linear
120 operator, insufficient Fourier modes exist, causing the system to approach a steady state solution. In this case,
121 the exponential time differencing methods integrate the system much more accurately than other methods since
122 the the exponential time differencing methods assume in their derivation that the solution varies slowly in time.

123 V.

124 6 Numerical Result

125 For the simulation tests, we choose two periodic initial conditions $u(x,0) = 2 \cos(x/4)$ and $u(x,0) = \sin(x/4) \cos(x/2)$

126 $u(x,0) = 2 \cos(x/4)$ and $u(x,0) = \sin(x/4) \cos(x/2)$
127 When evaluating the coefficients of the exponential time differencing and the exponential time differencing
128 Runge-Kutta methods via the "Cauchy integral" approach [6], we choose circular contours of radius $R =$
129 1 . Each contour is centered at one of the elements that are on the diagonal matrix of the linear part of the semi-
130 discretized model. We integrate the system (4.9) using fourth-order Runge Kutta exponential time differencing
131 scheme using $u_t + u u_x = \nu u_{xx} - u^3$

132 In the Figure 6, the solution appears more clear as a mesh plot and shows waves propagating, traveling
133 periodically in time and persisting without change of shape.

134 VI.

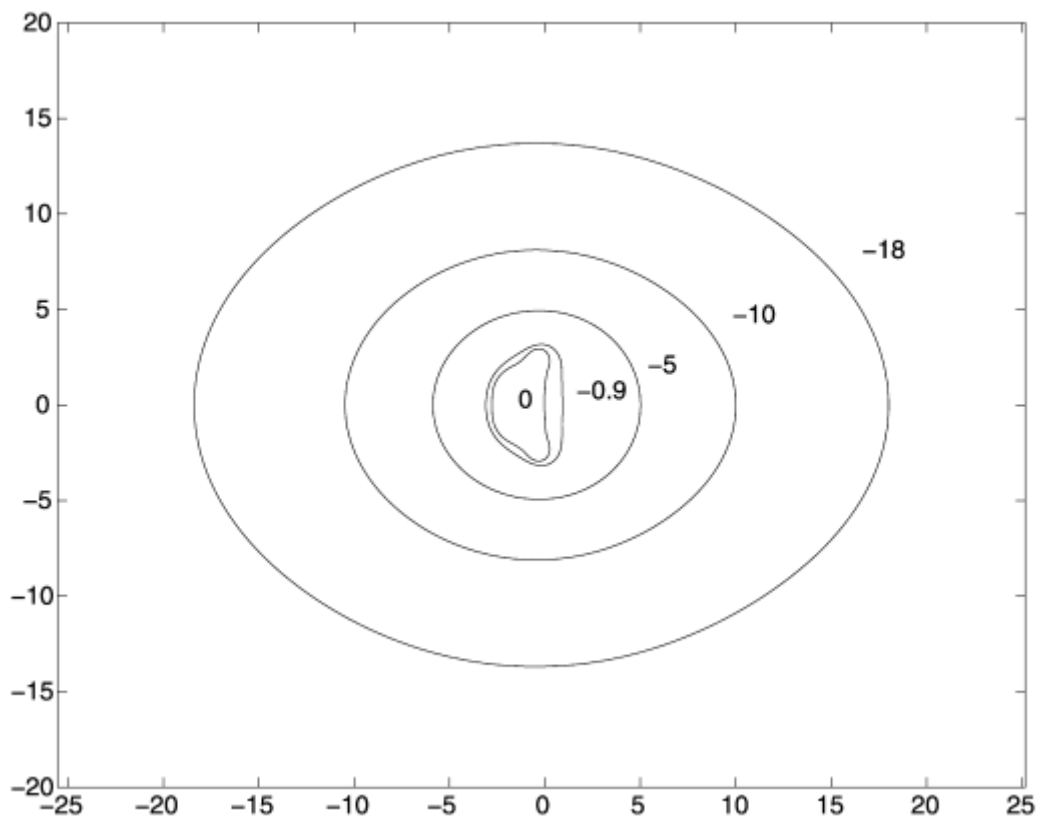
135 7 Conclusions

136 The proposers of the ETDRK schemes in [12] concluded that they are more accurate than other methods (standard
137 integrating factor techniques or linearly implicit schemes); they have good stability properties and are widely
138 applicable to nonlinear wave equations. However, Cox and Matthews were aware of the numerical instability
139 for the ETDRK4 method when computing the coefficients. Later, Kassar and Trefethen in [6] modified the
140 ETDRK4 method with very good results. In the opinion of these authors, the modified ETDRK4 is the best by
141 a clear margin compared with others methods. We have computed and studied the numerical stability function
142 of the ETDRK4 methods. In addition, we have applied this method to the Kuramoto-Sivashinsky equation,
143 achieving the excellent results that we have just mentioned. In order to achieve this, we applied Fourier spectral
144 approximation for the spatial discretization. For the simulation tests, we chose periodic boundary conditions
145 and applied Fourier spectral approximation for the spatial discretization. The equations can be used repeatedly
146 with necessary adaptations of the initial conditions.



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Figure 1: Figure 1 :Figure 2 :



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Figure 2: 2

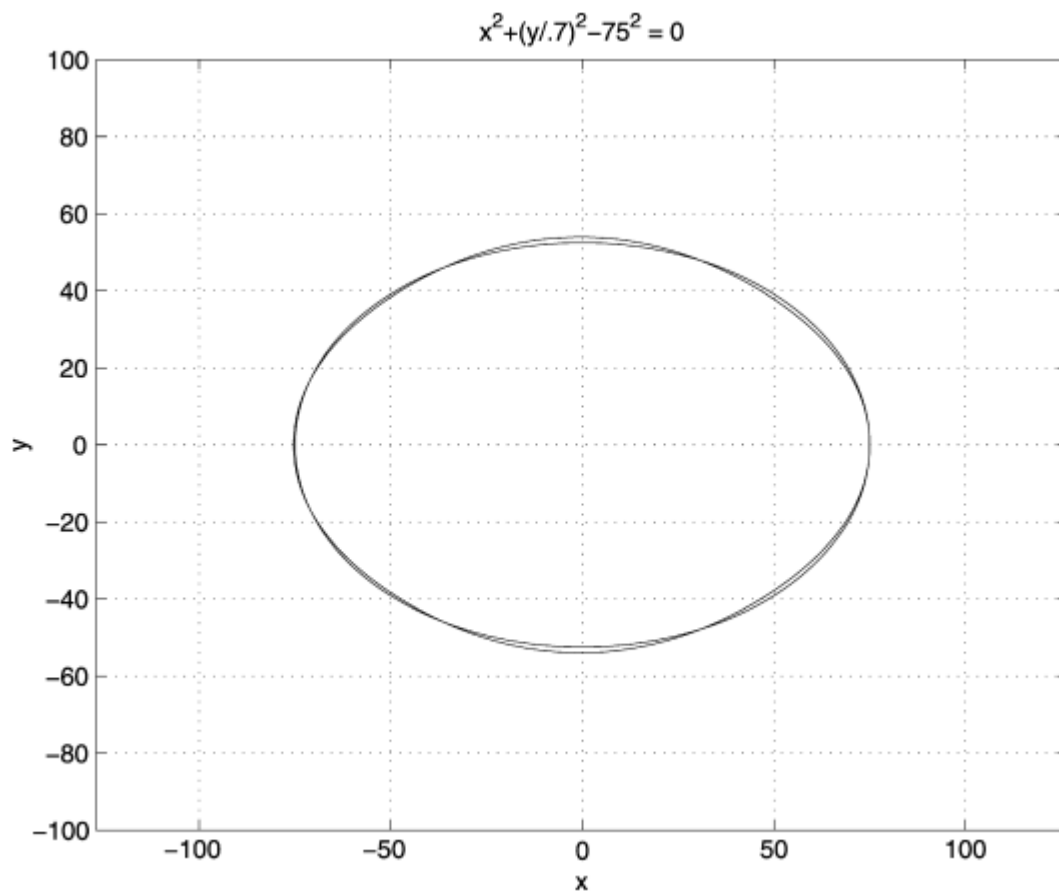


Figure 3:

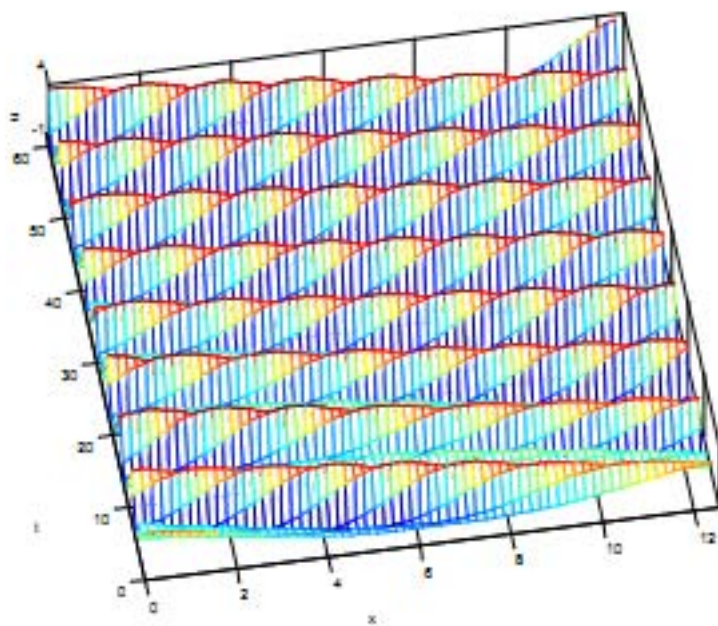
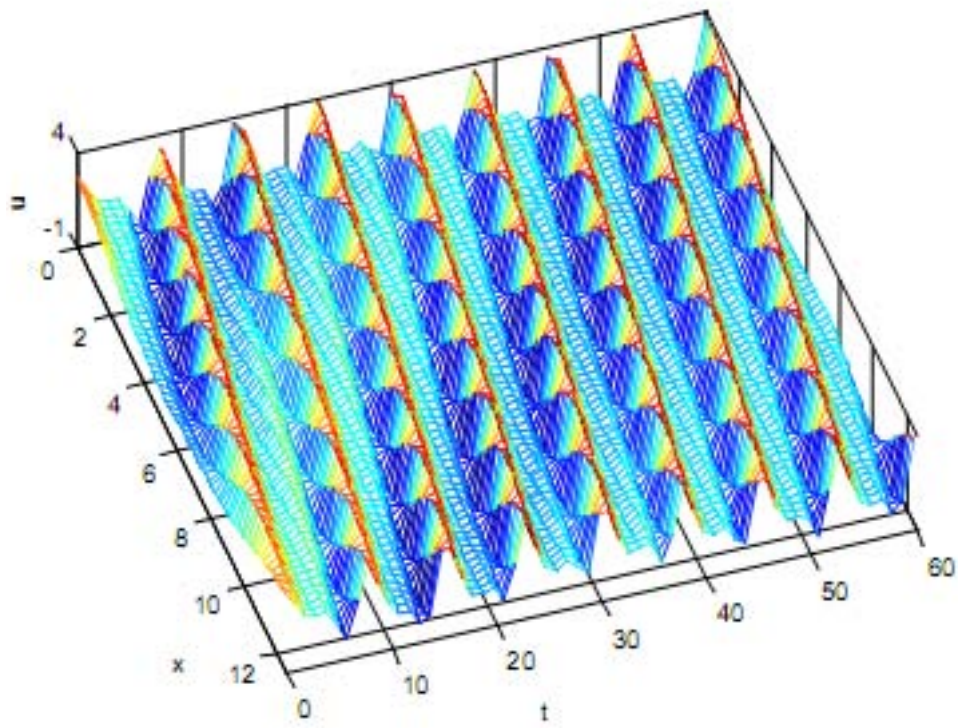


Figure 4: I



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Figure 5: 8)

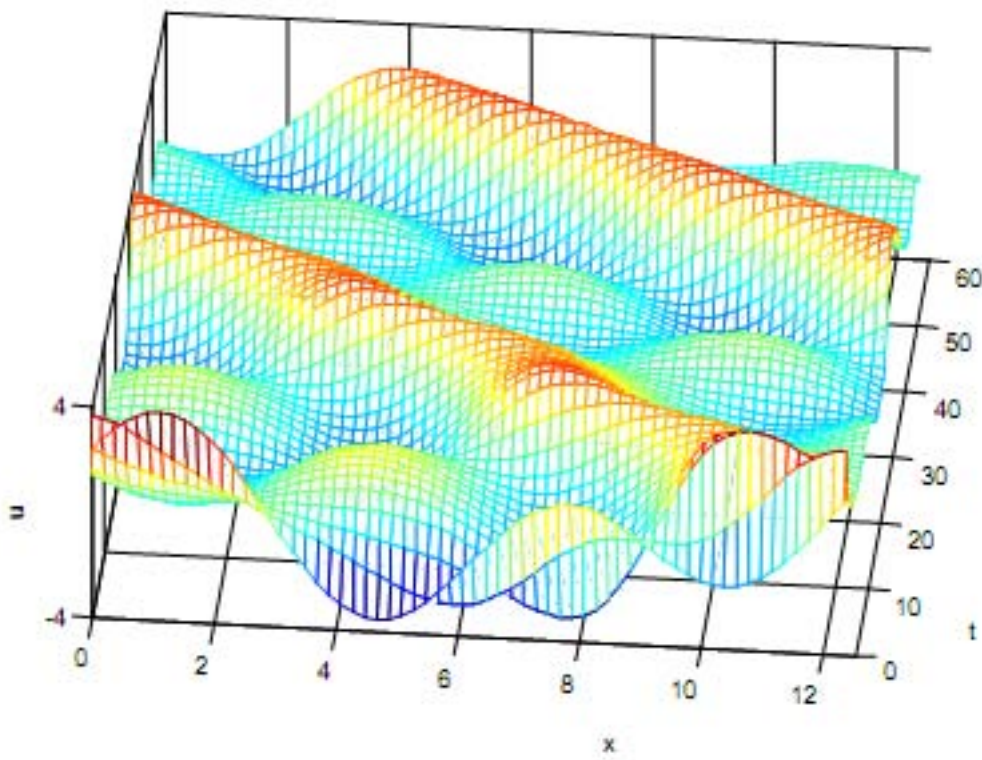
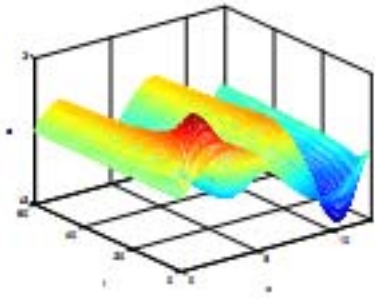


Figure 6:



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Figure 7: Figure 3 :<

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