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An Exponential Time Differencing Method for the Kuramoto-Sivashinsky Equation

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Gentian Zavalani

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I. Introduction

The spectral methods have been shown to be remarkably successful when solving time-dependent partial differential equations (PDEs).The idea is to approximate a solution $u(x,t)$ by a finite sum

$$\psi(x,t) = \sum_{k=0}^{N} \kappa_k(t)\varphi_k(x)$$

Where the function class $\varphi_k(x)$, $k = 0,1,2,...N$ will be trigonometric for x -periodic problems and, otherwise, an orthogonal polynomial of Jacobi type, with Chebyshev polynomial being the most important special case.To determine the expansion coefficients $\kappa_k(t)$, we will focus on the pseudospectral methods, where it is required that the coefficients make the residual equal zero at as many (suitably chosen) spatial points as possible.Three books [15,17] and [19] have been contributed to supplement the classic references [18] and [16]

When a time-dependent PDE is discretized in space with a spectral simulation, the result is a coupled system of ordinary differential equations (ODEs) in time: it is the notion of the method of lines and the resulting set of ODEs is stiff; the stiffness problem may be even exacerbated sometimes, for example, using Chebyshev polynomials.The linear terms are primarily responsible for the stiffness with rapid exponential decay of some modes (as with a dissipative PDE) or a rapid oscillation of some modes (as with a dispersive PDE).Therefore, for a time-dependent PDE which combines low-order nonlinear terms with higher-order linear terms it is desirable to use higher-order approximation in space and time.The outline of this paper is as follows. In Section2 we describe the ETDRK4 (Exponential Time Differencing fourth-order Runge–Kutta) method by Cox and Matthews in [12] and the modification proposed by Kassam and Trefethen in [6].We discuss the stability of the ETDRK4 method in Section 3. In Sections 4 and 5 we test the method for the Kuramoto-Sivashinsky equation in one space dimensions and, finally, In Sections 6 we summarize our conclusions.

II. Exponential Time Differencing Fourth-Order Runge–Kutta Method

The numerical method considered in this paper is an exponential time differencing (ETD) scheme. These methods arose originally in the field of computational electrodynamics [20]. Since then, they have recently received attention in [21] and [22], but the most comprehensive treatment, and in particular the ETD with Runge–Kutta time stepping, is in the paper by Cox and Matthews [12]. The idea of the ETD methods is similar to the method of the integrating factor (see, for example, [15] or [19]) we multiply both sides of a differential equation by some integrating factor, then we make a change of variable that allows us to solve the linear part exactly and, finally, we use a numerical method of our choice to solve the transformed nonlinear part.

When a time-dependent PDE in the form

$$u_t = \mathrm{L}\,u + \square\,(u,t) \qquad (2.1)$$

where $\mathcal{L}$ and $\mathbb{N}$ are the linear and nonlinear operators respectively, is discretized in space with a spectral method, the result is a coupled system of ordinary differential equations (ODEs),

$$u_t = Lu + N(u,t) \qquad (2.2)$$

Multiplying (2.2) by the terme e^{-Lt}, known as the integrating factor, gives

$$e^{-Lt}u_t - e^{-Lt}Lu = e^{-Lt}N(u,t) \qquad (2.3)$$

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and with the new variable $v = e^{-Lt}u$, we find the transformed equation

$$v_t = e^{-Lt} N(e^{Lt} v, t) \tag{2.4}$$

Where the linear term is gone; now we can use a time stepping method of our choice to advance in time. However, the integrating factor methods can also be a trap, for example, to model the formation and dynamics of solitary waves of the KdV equation (see Chapter 14 of[15]). A second drawback is the large error constant. In the derivation of the ETD methods, following [21], instead of changing the variable, we integrate (2.3) over a single time step of length, getting

$$u_{n+1} = e^{Lh} u_n + e^{Lh} \int_0^h e^{-Lh} N(u(t_n+\tau), t_n+\tau) d\tau \tag{2.5}$$

The various ETD methods come from how one approximates the integral in this expression. Cox and Matthews derived in [12] a set of ETD methods based on the Runge–Kutta time stepping, which they called ETDRK methods. In this paper we consider the ETDRK4 fourth-order scheme with the formulae

$$a_n = e^{Lh/2} u_n + L^{-1}(e^{Lh/2} - I) N(u_n, t_n)$$

$$b_n = e^{Lh/2} u_n + L^{-1}(e^{Lh/2} - I) N(a_n, u_n, t_n + \frac{h}{2})$$

$$b_n = e^{Lh/2} a_n + L^{-1}(e^{Lh/2} - I)\left[2N(b_n, t_n + \frac{h}{2}) - N(u_n, t_n) \right]$$

$$\begin{aligned} u_{n+1} = e^{Lh} u_n + h^{-2} L^{-3} \Big\{ & \left[-4I - hL + e^{Lh}(4I - 3hL + (hL)^2) \right] N(u_n, t_n) \\ & + 2[2I + hL + e^{Lh}(-2I + hL)] \left(N(a_n, t_n + \frac{h}{2}) + N(b_n, t_n + \frac{h}{2}) \right) \\ & + \left[-4I - 3hL - (hL)^2) + e^{Lh}(4I - hL) \right] N(c_n, t_n + \frac{h}{2}) \Big\} \end{aligned}$$

More detailed derivations of the ETD schemes can be found in [12].

Unfortunately, in this form ETDRK4 suffers from numerical instability when L has eigenvalues close to zero, because disastrous cancellation errors arise. Kassam and Trefethen have studied in [6] these instabilities and have found that they can be removed by evaluating a certain integral on a contour that is separated from zero. The procedure is basically to change the evaluation of the coefficients, which is mathematically equivalent to the original ETDRK4 scheme of [12], but in [23] it has been shown to have the effect of improving the stability of integration in time. Also, it can be easily implemented and the impact on the total computing time is small. In fact, we have always used this idea in our MATLAB© codes.

III. On the Stability of ETDRK4 Method

The stability analysis of the ETDRK4 method is as follows (see [21,24] or[12]). For the nonlinear ODE

$$\frac{du(t)}{dt} = cu(t) + F(u(t), t) \tag{3.1}$$

With $F(u(t), t)$ the nonlinear part, we suppose that there exists a fixed point u_0 this means that $cu_0 + F(u_0, t) = 0$. Linearizing about this fixed point, if $u(t)$ is the perturbation of u_0 and $\delta = F'(u_0, t)$ then

$$\frac{du(t)}{dt} = cu(t) + \delta u(t) \tag{3.2}$$

and the fixed point $u_0(t)$ is stable if $\mathrm{Re}(c+\delta) < 0$. The application of the ETDRK4 method to (3.2) leads to a recurrence relation involving u_n and u_{n+1}. Introducing the previous notation $x = \delta h$ and $y = ch$, and using the Mathematica© algebra package, we obtain the following amplification factor

$$\frac{u_{n+1}}{u_n} = r(x,y) = \ell_0 + \ell_1 x + \ell_2 x^2 + \ell_3 x^3 + \ell_4 x^4 \tag{3.3}$$

where

$$\ell_0 = e^y$$

$$\ell_1 = \frac{-4}{y^3} + \frac{8e^{\frac{y}{2}}}{y^3} - \frac{8e^{\frac{3y}{2}}}{y^3} + \frac{4e^{2y}}{y^3} - \frac{1}{y^2} + \frac{4e^{\frac{y}{2}}}{y^2} - \frac{6e^{y}}{y^2} + \frac{4e^{\frac{3y}{2}}}{y^2} - \frac{e^{2y}}{y^2}$$

$$\ell_2 = \frac{-8}{y^4} + \frac{16e^{\frac{y}{2}}}{y^4} - \frac{16e^{\frac{3y}{2}}}{y^4} + \frac{8e^{2y}}{y^4} - \frac{5}{y^3} + \frac{12e^{\frac{y}{2}}}{y^3} - \frac{10e^{y}}{y^3} + \frac{4e^{\frac{3y}{2}}}{y^3} - \frac{e^{2y}}{y^3} - \frac{1}{y^2} + \frac{4e^{\frac{y}{2}}}{y^2} - \frac{3e^{y}}{y^3},$$

$$\ell_3 = \frac{4}{y^5} - \frac{16e^{\frac{y}{2}}}{y^5} + \frac{16e^{y}}{y^5} + \frac{8e^{\frac{3y}{2}}}{y^5} - \frac{20e^{2y}}{y^5} + \frac{8e^{5\frac{y}{2}}}{y^5} - \frac{2}{y^4} + \frac{-10e^{\frac{y}{2}}}{y^4} + \frac{16e^{y}}{y^4} - \frac{12e^{\frac{3y}{2}}}{y^4} + \frac{6e^{2y}}{y^4} - \frac{2e^{5\frac{y}{2}}}{y^4}$$
$$- \frac{2e^{\frac{y}{2}}}{y^3} + \frac{4e^{y}}{y^3} - \frac{2e^{3\frac{y}{2}}}{y^3}$$

$$\ell_4 = \frac{8}{y^6} - \frac{24e^{\frac{y}{2}}}{y^6} + \frac{16e^{y}}{y^6} + \frac{16e^{\frac{3y}{2}}}{y^5} - \frac{24e^{2y}}{y^6} + \frac{8e^{5\frac{y}{2}}}{y^6} + \frac{6}{y^5} + \frac{-18e^{\frac{y}{2}}}{y^5} + \frac{20e^{y}}{y^5} - \frac{12e^{\frac{3y}{2}}}{y^5} + \frac{6e^{2y}}{y^5} + \frac{2e^{5\frac{y}{2}}}{y^5}$$
$$+ \frac{2}{y^4} - \frac{6e^{\frac{y}{2}}}{y^4} + \frac{6e^{y}}{y^4} - \frac{2e^{3\frac{y}{2}}}{y^4}$$

An important remark: computing $\ell_0, \ell_1, \ell_2, \ell_3, \ell_4$ by the above expressions suffers from numerical instability for y close to zero. Because of that, for small y, instead of them, we will use their asymptotic expansions.

$$\ell_1 = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{13}{320}y^4 + \frac{7}{960}y^5 + O(y^6)$$

$$\ell_2 = \frac{1}{2} + \frac{1}{2}y + \frac{1}{4}y^2 + \frac{247}{2880}y^3 + \frac{131}{5760}y^4 + \frac{479}{96768}y^5 + O(y^6)$$

$$\ell_3 = \frac{1}{6} + \frac{1}{6}y + \frac{61}{720}y^2 + \frac{1}{36}y^3 + \frac{1441}{241920}y^4 + \frac{67}{120960}y^5 + O(y^6)$$

$$\ell_4 = \frac{1}{24} + \frac{1}{32}y + \frac{7}{640}y^2 + \frac{19}{11520}y^3 - \frac{311}{64512}y^4 - \frac{479}{860160}y^5 + O(y^6)$$

We make two observations:

- As $y \mapsto 0$ y, our approximation becomes

  $$r(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

  which is the stability function for all the 4-stage Runge–Kutta methods of order four.

- Because c and δ may be complex, the stability region of the ETDRK4 method is four-dimensional and therefore quite difficult to represent. Unfortunately, we do not know any expression for $|r(x,y)| = 1$ we will only be able to plot it. The most common idea is to study it for each particular case; for example, assuming c to be fixed and real in [21] or that both c and δ are pure imaginary numbers in [24].

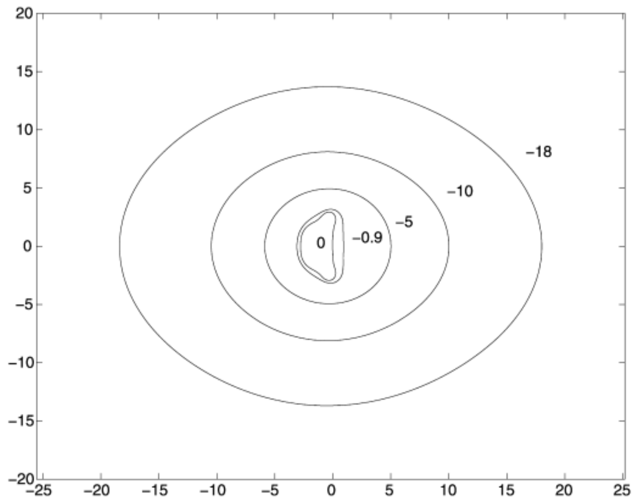

Figure 1 : Boundary of stability regions for several negativey

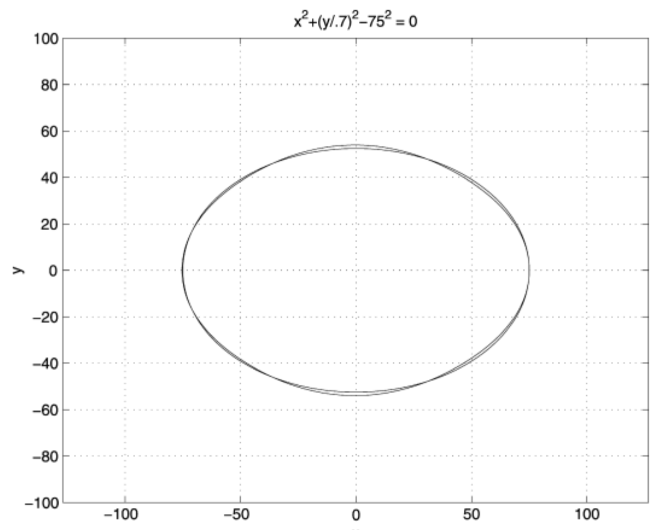

Figure 2 : Experimental boundaries and ellipse for y= −75

For dissipative PDEs with periodic boundary conditions, the scalars c that arise with a Fourier spectral method are negative. Let us take for example Burger's equation

$$u_t = \varepsilon u_{xx} - (\frac{1}{2}u^2)_x \quad x \in [-\pi, \pi] \text{ where } 0 < \varepsilon \ll 1 \quad (3.4)$$

Transforming it to the Fourier space gives

$$\tilde{u}_t = -\varepsilon\zeta^2\tilde{u} - \frac{i\zeta}{2}\tilde{u}^2 \quad \forall\zeta \qquad (3.5)$$

where $\forall\zeta$ is the Fourier wave-number and the coefficients $c = -\varepsilon\zeta^2 < 0$, span over a wide range of values when all the Fourier modes are considered. For high values of $|\zeta|$ the solutions are attracted to the slow manifold quickly because $c < 0$ and $|c| << 1$.

In *Figure.1* we draw the boundary stability regions in the complex plane x for $y=0,-0.9,-5,-10,-18$. When the linear part is zero ($y=0$), we recognized the stability region of the fourth-order Runge–Kutta methods and, as $y \mapsto -\infty$, the region grows. Of course, these regions only give an indication of the stability of the method.

In fact, for $y < 0$, $|y| << 1$ the boundaries that are observed approach to ellipses whose parameters have been fitted numerically with the following result.

$$(\mathrm{Re}(x))^2 + \left(\frac{\mathrm{Im}(x)}{0.7}\right)^2 = y^2 \qquad (3.6)$$

In *Figure 2* we draw the experimental boundaries and the ellipses (3.6) with $y = -75$ .The spectrum of the linear operator increases as ζ^2 , while the eigenvalues of the linearization of the nonlinear part lay on the imaginary axis and increase as ζ . On the other hand, according to (3.6), when $\mathrm{Re}(x)=0$, the intersection with the imaginary axis $\mathrm{Im}(x)$ increases as $|y|$, i.e., as ζ^2 .Since the boundary of stability grows faster than , the ETDRK4 method should have a very good behavior to solve Burger's equation, which confirms the results of paper [6].

IV. The Kuramoto-Sivashinsky Equation

The Kuramoto-Sivashinsky equation (K-S), is one of the simplest PDEs capable of describing complex behavior in both time and space. This equation has been of mathematical interest because of its rich dynamical properties. In physical terms, this equation describes reaction diffusion problems, and the dynamics of viscous-fuid films flowing along walls.

Kuramoto-Sivashinsky equation in one space dimension can be written

$$\frac{\partial u(x,t)}{\partial t}=-u(x,t)\frac{\partial u(x,t)}{\partial x}-\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\partial^4 u(x,t)}{\partial x^4} \quad (4.1)$$

or in form

$$u_t+\nabla^4 u+\nabla^2 u+\left|\nabla\ u\right|^2/2=0$$

Equation (4.1) can be written in integral form if we introduce

$$u(x,t)=\frac{\partial\zeta(x,t)}{\partial t}$$

then

$$\frac{\partial\zeta(x,t)}{\partial t}=-\frac{1}{2}\left(\frac{\partial\zeta(x,t)}{\partial x}\right)^2-\frac{\partial^2\zeta(x,t)}{\partial x^2}-\frac{\partial^4\zeta(x,t)}{\partial x^4} \quad (4.2)$$

The Kuramoto-Sivashinsky equation with 2L periodic boundary conditions in Fourier space can be written as follows

$$u_j(x)=\sum_{k=0}^{N_\tau-1}\tilde{u}_k\ e^{\frac{ik\pi x}{L}} \quad (4.3)$$

$$u_j(x,t)=\sum_{k=0}^{N_\tau-1}\tilde{u}_k\, e^{\frac{ik\pi x}{L}} \quad (4.4)$$

$$\frac{\partial u_j(x,t)}{\partial t}=\sum_{k=0}^{N_\tau-1}\frac{d\tilde{u}_k}{dt}\tilde{u}_k\, e^{\frac{ik\pi x}{L}} \quad (4.5)$$

$$\frac{\partial u_j(x,t)}{\partial x}=\sum_{k=0}^{N_\tau-1}\frac{ik\pi}{L}\tilde{u}_k\, e^{\frac{ik\pi x}{L}} \quad (4.6)$$

$$\frac{\partial^2 u_j(x,t)}{\partial x^2}=\sum_{k=0}^{N_\tau-1}\left(\frac{ik\pi}{L}\right)^2\tilde{u}_k\, e^{\frac{ik\pi x}{L}} \quad (4.7)$$

$$\frac{\partial^4 u_j(x,t)}{\partial x^4}=\sum_{k=0}^{N_\tau-1}\left(\frac{ik\pi}{L}\right)^4\tilde{u}_k\, e^{\frac{ik\pi x}{L}} \quad (4.8)$$

If we substitute (4.3), (4.4), (4.5), (4.6), (4.7), (4.8) into (4.1) we get

$$\frac{\partial u(x,t)}{\partial t}=-u(x,t)\frac{\partial u(x,t)}{\partial x}-\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\partial^4 u(x,t)}{\partial x^4}\Leftrightarrow\frac{\partial u_j(x,t)}{\partial t}=-u_j(x,t)\frac{\partial u_j(x,t)}{\partial x}-\frac{\partial^2 u_j(x,t)}{\partial x^2}-\frac{\partial^4 u_j(x,t)}{\partial x^4}$$

$$\Leftrightarrow\sum_{k=0}^{N_\tau-1}\frac{d\tilde{u}_k}{dt}e^{\frac{ik\pi x}{L}}=-\left[\left(\sum_{k=0}^{N_\tau-1}\tilde{u}_k\, e^{\frac{ik\pi x}{L}}\right)*\left(\sum_{k=0}^{N_\tau-1}\frac{ik\pi}{L}\tilde{u}_k\, e^{\frac{ik\pi x}{L}}\right)\right]-\left[\sum_{k=0}^{N_\tau-1}\left(\frac{ik\pi}{L}\right)^2\tilde{u}_k\, e^{\frac{ik\pi x}{L}}\right]-\left[\sum_{k=0}^{N_\tau-1}\left(\frac{ik\pi}{L}\right)^4\tilde{u}_k\, e^{\frac{ik\pi x}{L}}\right]$$

By simplifying and note that

$$\tilde{u}_k=\frac{1}{N_\tau}\sum_{j=0}^{N_\tau-1}u_j e^{-2ijk/N_\tau},\ (i)^2=-1,(i)^4=1\ \ x_j=jh,h=\frac{2L}{N_\tau}$$

Equation (4.1) can be written as fellows

$$\frac{\partial\tilde{u}_k(t)}{\partial t}=\left(k^2-k^4\right)\tilde{u}_k(t)-\frac{ik}{2}\varpi_k \quad (4.9)$$

Where $\varpi_k = \frac{1}{2L}\int_{-L}^{L} u^2(x,t) e^{\frac{ik\pi x}{L}} dx = \frac{1}{N_\tau}\sum_{j=0}^{N_\tau-1} u^2{}_j(x,t) e^{-2ijk/N_\tau} = FFT\left[u(t)^2\right]$

In final form will be

$$\frac{\partial \tilde{u}_k(t)}{\partial t} = \left(k^2 - k^4\right)\tilde{u}_k(t) - \frac{ik}{2}\varpi_k \qquad (4.9)$$

Equation has strong dissipative dynamics, which arise from the fourth order dissipation $\frac{\partial^4 u(x,t)}{\partial x^4}$ term that provides damping at small scales. Also, it includes the mechanisms of a linear negative diffusion $\frac{\partial^2 u(x,t)}{\partial x^2}$ term, which is responsible for an instability of modes with large wavelength,i.e small wave-numbers. The nonlinear advection/steepening $u(x,t)\frac{\partial u(x,t)}{\partial x}$ term in the equation transforms energy between large and small scales.

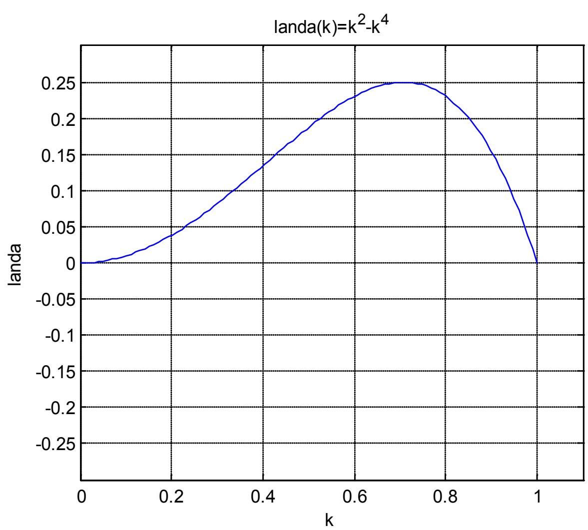

Figure 3 : The growth rate for perturbations of the form $e^{\lambda t}e^{ikx}$ to the zero solution of the Kuramoto-Sivashinsky (K-S) equation

The zero solution of the K-S equation is linearly unstable (the growth rate $\lambda(k) > 0$ for perturbations of the form $e^{\lambda t}e^{ikx}$ to modes with wave-numbers $|k| = \left|\frac{2\pi}{\vartheta}\right| < 1$ for a wavelength ϑ and is damped for modes with $|k| > 1$ see *Figure3:* these modes are coupled to each other through the non-linear term.

The stiffness in the system (4.9) is due to the fact that the diagonal linear operator $\left(k^2 - k^4\right)$,with the elements , has some large negative real eigenvalues that represent decay, because of the strong dissipation, on a time scale much shorter than that typical of the nonlinear term. The nature of the solutions to the the Kuramoto-Sivashinsky equation varies with the system size of linear operator. For large size of linear operator, enough unstable Fourier modes exist to make the system chaotic. For small size of linear operator, insuffcient Fourier modes exist, causing the system to approach a steady state solution. In this case, the exponential time differencing methods integrate the system much more accurately than other methods since the the exponential time differencing methods assume in their derivation that the solution varies slowly in time.

V. Numerical Result

For the simulation tests, we choose two periodic initial conditions

$$u_1(x) = e^{\cos\left(\frac{x}{2}\right)}, x \in [0, 4\pi]$$

$$u_2(x) = 1.7\cos\left(\frac{x}{2}\right) + 0.1\sin\left(\frac{x}{2}\right) + 0.6\cos(x) + 2.4\sin(x), x \in [0, 4\pi]$$

When evaluating the coefficients of the exponential time differencing and the exponential time differencing Runge- Kutta methods via the "Cauchy integral" approach [5],[6] we choose circular contours of radius R = 1. Each contour is centered at one of the elements that are on the diagonal matrix of the linear part of the semi-discretized model. We integrate the system (4.9) using fourth-order Runge Kutta exponential time differencing scheme using $N_\tau = 64$ with time-step size $\Delta t = 2e-10$.

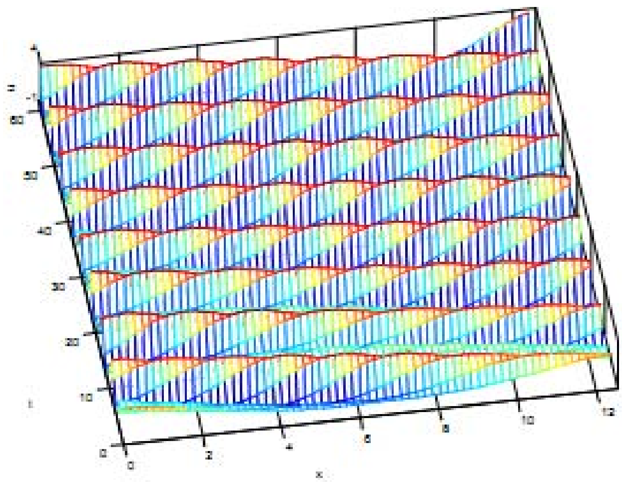

Figure 4 : Time evolution of the numerical solution of the Kuramoto-Sivashinsky up to t = 60 with the initial condition $u_1(x) = e^{\cos\left(\frac{x}{2}\right)}, x \in [0, 4\pi]$

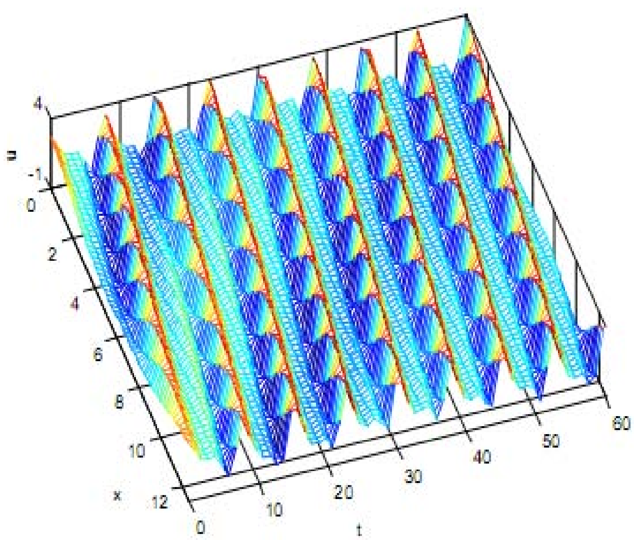

Figure 5 : Time evolution of the numerical solution of the Kuramoto-Sivashinsky up to t = 60 with the initial condition $u_1(x) = e^{\cos\left(\frac{x}{2}\right)}, x \in [0, 4\pi]$

The solution, in the *Figure 5* with the initial condition $u_1(x) = e^{\cos\left(\frac{x}{2}\right)}, x \in [0, 4\pi]$ with $N_\tau = 64$ and time-step size $\Delta t = 2e-10$, appears as a mesh plot and shows waves propagating, traveling periodically in time and persisting without change of shape.

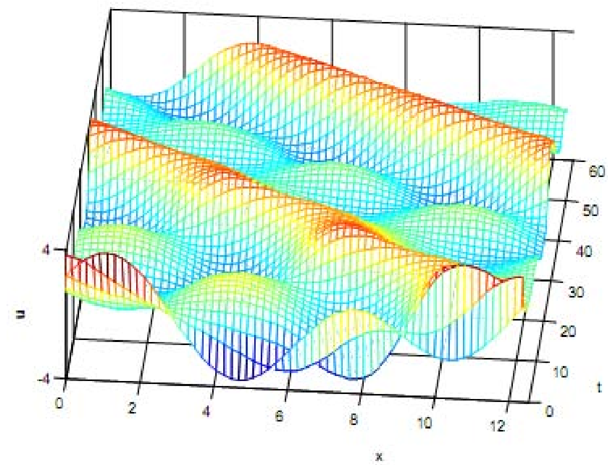

Figure 6 : Time evolution of the numerical solution of the Kuramoto-Sivashinsky up to t = 60 with the initial condition $u_2(x) = 1.7\cos\left(\frac{x}{2}\right) + 0.1\sin\left(\frac{x}{2}\right) + 0.6\cos(x) + 2.4\sin(x), x \in [0, 4\pi]$

In the *Figure 6* with the initial condition $u_2(x) = 1.7\cos\left(\frac{x}{2}\right) + 0.1\sin\left(\frac{x}{2}\right) + 0.6\cos(x) + 2.4\sin(x), x \in [0, 4\pi]$ with $N_\tau = 64$ and time-step size $\Delta t = 2e-10$, the solution appears as a mesh plot and shows waves propagating, traveling periodically in time and persisting without change of shape.

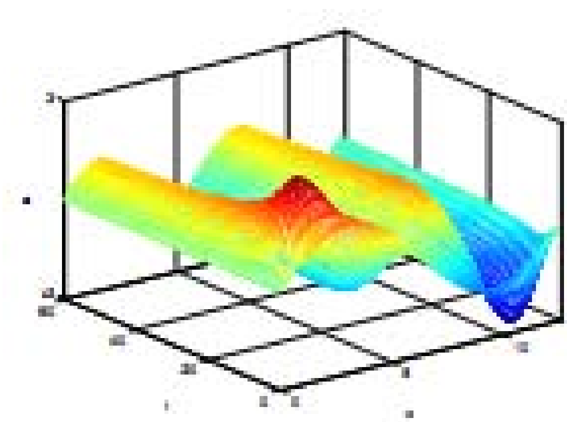

Figure 7 : Time evolution of the numerical solution of the Kuramoto-Sivashinsky up to t = 60 with the initial condition $u_1(x) = \sin(x/2), x \in [0, 4\pi]$

In the *Figure 7* with the initial condition $u_1(x) = \sin(x/2), x \in [0, 4\pi]$ with $N_\tau = 64$ and time-step size $\Delta t = 2e-10$, the solution appears more clear as a mesh plot and shows waves propagating, traveling periodically in time and persisting without change of shape.

VI. Conclusions

The proposers of the ETDRK schemes in [12] concluded that they are more accurate than other methods (standard integrating factor techniques or linearly implicit schemes); they have good stability properties and are widely applicable to nonlinear wave equations. However, Cox and Matthews were aware of the numerical instability for the ETDRK4 method when computing the coefficients. Later, Kassan and Trefethen in [6] modified the ETDRK4 method with very good results. In the opinion of these authors, the modified ETDRK4 is the best by a clear margin compared with others methods.We have computed and studied the numerical stability function of the ETDRK4 methods. In addition, we have applied this method to the Kuramoto-Sivashinsky equation, achieving the excellent results that we have just mentioned. In order to achieve this, we applied Fourier spectral approximation for the spatial discretization. .For the simulation tests, we chose periodic boundary conditions and applied Fourier spectral approximation for the spatial discretization. . The equations can be used repeatedly with necessary adaptations of the initial conditions.

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