Global Journal of Researches in Engineering: I
Numerical Methods
Volume 14 Issue 1 Version 1.0 Year 2014
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4596 & Print ISSN: 0975-5861

Fourier Spectral Methods for Numerical Solving of the Kuramoto-Sivashinsky Equation

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GJRE-I Classification : FOR Code: 010106, 230116

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I. Introduction

Fourier analysis occurs in the modeling of time-dependent phenomena that are exactly or approximately periodic. Examples of this include the digital processing of information such as speech; the analysis of natural phenomen such as earthquakes; in the study of vibrations of spherical, circular or rectangular structures; and in the processing of images. In a typical case, Fourier spectral methods write the solution to the partial differential equation as its Fourier series. Fourier series decomposes a periodic real-valued function of real argument into a sum of simple oscillating trigonometric functions ($sines, cosines$), that can be recombined to obtain the original function. Substituting this series into the partial differential equation gives a system of ordinary differential equations for the time-dependent coefficients of the trigonometric terms in the series then we choose a time-stepping method to solve those ordinary differential equations

II. Fourier Series

The Fourier series of a smooth and periodic real-valued function $f(x) \in [0; 2L]$
with period $2L$ is

$$\mathrm{f(x)} = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \qquad (1)$$

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Since the basis functions $\sin(\frac{n\pi x}{L})$ and $\cos(\frac{n\pi x}{L})$ are orthogonal the coffiecients are given by

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 0,1,2.... \qquad (2)$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2.... \qquad (3)$$

Fourier series can be expressed neatly in complex form as follows

$$\mathrm{f(x)} = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left(e^{\frac{in\pi x}{L}} + e^{\frac{-in\pi x}{L}}\right) + \frac{b_n}{2i}\left(e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}}\right)\right] \qquad (4)$$

If we define

$$c_0 = \frac{a_o}{2}, c_n = \frac{a_n - ib_n}{2}, c_{-n} = \frac{a_n + ib_n}{2} \qquad (5)$$

Then

$$\mathrm{f(x)} = \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{L}} \qquad (6)$$

where the coefficients c_n can be determined from the formulas of a_n and b_n as

$$c_n = \frac{1}{2L}\int_0^{2L} f(x) e^{\frac{in\pi x}{L}} dx \qquad (7)$$

III. Discrete Fourier Transform

In many applications, particularly in analyzing of real situations, the function $f(x)$ to be approximated is known only on a discrete set of “sampling points" of x. Hence, the integral (7) cannot be evaluated in a closed form and Fourier analysis cannot be applied directly. It then becomes necessary to replace continuous Fourier analysis by a discrete version of it. The linear discrete Fourier transform of a periodic (discrete) sequence of complex values $u_0, u_0, \dots, u_{N_\tau - 1}$ with period u_{N_τ}, is a sequence of periodic complex values $\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_{N_\tau - 1}$ defined by

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$$\tilde{u}_k = \frac{1}{N_\tau}\sum_{j=0}^{N_\tau-1} u_j e^{\frac{-2ij\pi k}{N_\tau}} \quad (8)$$

The linear inverse transformation is

$$u_j = \sum_{k=0}^{N_\tau-1} \tilde{u}_k e^{\frac{2ij\pi k}{N_\tau}} \quad (9)$$

The most obvious application of discrete Fourier analysis consists in the numerical calculation of Fourier coefficients. Suppose we want to approximate a real-valued periodic function $f(x)$ defined on the interval $[0;2L]$ that is sampled with an even number N_τ of grid points

$$x_j = jh \qquad h = \frac{2L}{N_\tau} \qquad j = 0,1,\ldots,N_\tau - 1 \quad (10)$$

by it is Fourier series. First we compute approximate values of the Fourier coefficients

$$\tilde{c}_k \approx \frac{1}{N_\tau}\sum_{j=0}^{N_\tau-1} f(x) e^{\frac{-2ij\pi k}{N_\tau}} \quad (11)$$

Because the discrete Fourier transform and its inverse exhibit periodicity with period N_τ , i.e. $\tilde{u}_k + N_\tau = \tilde{u}_k$ (this property results from the periodic nature of $e^{\frac{2\pi ijk}{N_\tau}}$), it makes no sense to use more than N_τ terms in the series, and it suffices to calculate one full period. The Fourier series formed with the approximate coefficients is

$$\tilde{f}(x) \approx \sum_{k=-N_\tau/2+1}^{N_\tau/2} \tilde{c}_k e^{\frac{-2ij\pi k}{N_\tau}} \quad (11')$$

The function $\tilde{f}(x)$ not only approximates, but actually interpolates $f(x)$ at the sampling grid points x_j

In matrix form, the discrete Fourier transform (8) can be written as

$$\tilde{u}_k = \frac{1}{N_\tau} M_{kj} u_j \quad k, j = 0,1,\ldots N_\tau - 1 \quad (12)$$

Where $M_{kj} = \omega^{kj}$ and $\omega = e^{\frac{-2i\pi}{N_\tau}}$

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \ldots & \omega^{N\tau-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \ldots & \omega^{2(N\tau-1)} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & \omega^{N\tau-1} & \omega^{2(N\tau-1)} & \omega^{3(N\tau-1)} & \ldots & \omega^{(N\tau-1)(N\tau-1)} \end{pmatrix}$$

Similarly, the inverse discrete Fourier transform has the form

$$u_j = M^*_{kj}\tilde{u}_k \quad k, j = 0,1,\ldots N_\tau - 1 \quad (13)$$

Where $M^*_{kj} = \omega^{*kj}$ and $\omega^* = e^{\frac{2i\pi}{N_\tau}}$ where ω^* is complex conjugate of ω

$$M^* = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega^* & \omega^{*2} & \omega^{*3} & \ldots & \omega^{*N\tau-1} \\ 1 & \omega^{*2} & \omega^{*4} & \omega^{*6} & \ldots & \omega^{*2(N\tau-1)} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & \omega^{*N\tau-1} & \omega^{*2(N\tau-1)} & \omega^{*3(N\tau-1)} & \ldots & \omega^{*(N\tau-1)(N\tau-1)} \end{pmatrix}$$

The FFT algorithm reduces the computational work required to carry out a discrete Fourier transform by reducing the number of multiplications and additions of (13), computational time is reduced from $O(N^2{}_\tau)$ to $O(N_\tau \log N_\tau)$.

To apply spectral methods to a partial differential equation we need to evaluate derivatives of functions. Suppose that we have a periodic real-valued function $f(x) \in [0;2L]$ with period $2L$ that is discretized with an even number N_τ of grid points, so that the grid size $h = \frac{2L}{N_\tau}$.The complex form of the Fourier series representation of $f(x)$ is

$$\tilde{f}(x) \approx \sum_{k=-N_\tau/2+1}^{N_\tau/2} \tilde{c}_k e^{\frac{i\pi kx}{L}}$$

At $k = \frac{N_\tau}{2}$ the above series gives a term $\tilde{c}_{N_\tau/2} e^{\frac{i\pi N_\tau x}{2L}}$, which alternates between $\pm\tilde{c}_{N_\tau/2}$ at the grid point $x_j = jh$, $j = 0,1,....,N_\tau - 1$, and since it cannot be differentiated, we should set its derivative to be zero at the grid points. The numerical derivatives of the function $f(x)$ can be illustrated as a matrix multiplication.For the first derivative, we multiply the Fourier coefficients (11) by the corresponding differentiation matrix for an even number N_τ of grid points.

$$\Lambda_1 = Diag\left(0,1,2,3....,\frac{N_\tau}{2}-1,0,-\left(\frac{N_\tau}{2}-1\right),....,-3-2-1\right)\frac{i\pi}{L}$$

This matrix has non-zero elements only on the diagonal. For an odd number N_τ of grid points the differentiation matrix corresponding to the first derivative is diagonal with elements.

$$\left(0,1,2,3....,\frac{N_\tau}{2}-1,0,-\left(\frac{N_\tau}{2}-1\right),....,-3-2-1\right)\frac{i\pi}{L}$$

Then, we compute an inverse discrete Fourier transform using $(11')$ to return to the physical space and deduce the first derivative of f (x) on the grid. Similarly, taking the second derivative corresponds to the multiplication of the Fourier coefficients (11) by the corresponding differentiation matrix for an even number N_τ of grid points.

$$\Lambda_2 = Diag\left(0,1,4,9....,\left(\frac{N_\tau}{2}-1\right)^2,\left(\frac{N_\tau}{2}\right)^2,\left(\frac{N_\tau}{2}-1\right)^2,.....,9,4,1\right)\left(\frac{i\pi}{L}\right)^2$$

In general, in case of an even number N_τ of grid points approximating the m-th numerical derivatives of a grid function $f(x)$ corresponds to the multiplication of the Fourier coefficients (11) by the corresponding differentiation matrix which is diagonal with elements $\left(\frac{ik\pi}{L}\right)^m$

for

$$k = 0,1,2,3....,\frac{N_\tau}{2}-1,\frac{N_\tau}{2},-\left(\frac{N_\tau}{2}-1\right),....,-3-2-1$$

with the exception that for odd derivatives we set the derivative of the highest mode $k = \frac{N_\tau}{2}$ to be zero.

IV. Exponential Time Differencing

The family of exponential time differencing schemes. This class of schemes is especially suited to semi-linear problems which can be split into a linear part which contains the stiffest part of the dynamics of the problem, and a nonlinear part, which varies more slowly than the linear part. Exponential time differencing schemes are time integration methods that can be efficiently combined with spatial approximations to provide accurate smooth solutions for stiff or highly oscillatory semi-linear partial differential equations.In this paper I will present the derivation of the explicit Exponential time differencing schemes for arbitrary order following the approach in [12],[2],[4] and presents the explicit Runge-Kutta versions of these schemes constructed by Cox and Matthews [12].

We consider for simplicity a single model of a stiff ordinary differential equation

$$\frac{du(t)}{dt} = cu(t) + F(u(t),t) \quad (e)$$ where $F(u(t),t)$ is the nonlinear forcing term.

To derive the s-step Exponential time differencing schemes, we multiply through by the integrating factor e^{-ct} and then integrate the equation over a single time step from $t = t_n$ to $t = t_{n+1} = t_n + \Delta t$ to obtain.

$$u(t_{n+1}) = u(t_n)e^{c\Delta t} + e^{c\Delta t} * \int_0^{\Delta t} F(u(t_n+\tau), t_n+\tau)d\tau \quad (e_1)$$

The next step is to derive approximations to the integral in equation (e_1). This procedure does not introduce an unwanted fast time scale into the solution and the schemes can be generalized to arbitrary order.

If we apply the Newton Backward Difference Formula, we can write a polynomial approximation to $F(u(t_n+\tau), t_n+\tau)$ in the form

$$F(u(t_n+\tau), t_n+\tau) \approx G(t_n, t) = \sum_{m=0}^{s-1} (-1)^m \binom{-\tau/\Delta t}{m} * \nabla^m G_n(t_n) \approx$$

$$\approx \sum_{m=0}^{s-1} (-1)^m \binom{-\tau/\Delta t}{m} * \underbrace{\sum_{k=0}^{m} (-1)^m \binom{m}{k} F(u(t_{n-k}), t_{n-k})}_{\nabla^m G_n(t_n)} \quad (e_2)$$

$$k!\binom{m}{k} = (m-1)(m-2)\ldots(m-k+1), m = 1, \ldots, s-1 \text{ note that } 0!\binom{m}{0} = 1$$

If we substitute (e_2) into (e_1) we get

$$u(t_{n+1}) = u(t_n)e^{c\Delta t} + e^{c\Delta t} * \int_0^{\Delta t} \sum_{m=0}^{s-1} (-1)^m \binom{-\tau/\Delta t}{m} * \nabla^m G_n(t_n) d\tau$$

$$u(t_{n+1}) - u(t_n)e^{c\Delta t} = \Delta t \sum_{m=0}^{s-1} (-1)^m * \int_0^1 e^{c\Delta t(1-\tau/\Delta t)} \binom{-\tau/\Delta t}{m} * \nabla^m G_n(t_n) d(\tau/\Delta t) \quad (e_3)$$

We will indicate the integral by $\vartheta_m = \int_0^1 e^{c\Delta t(1-\tau/\Delta t)} \binom{-\tau/\Delta t}{m} d(\tau/\Delta t) \quad (e_4)$

If we substitute (e_2) and (e_4) into (e_3) we get

$$u(t_{n+1}) = u(t_n)e^{c\Delta t} + \Delta t \sum_{m=0}^{s-1} (-1)^m * \vartheta_m * \sum_{k=0}^{m} (-1)^m \binom{m}{k} F(u(t_{n-k}), t_{n-k}) \quad (e_5)$$

Which represent the general generating formula of the exponential time differencing schemes of order s

The first-order exponential time differencing scheme is obtained by setting s=1

$$u_{n+1} = u_n e^{c\Delta t} + (e^{c\Delta t} - 1)F_n / c$$

The second-order exponential time differencing scheme is obtained by setting s=2

$$u_{n+1} = u_n e^{c\Delta t} + \{((c\Delta t + 1)e^{c\Delta t} - 2c\Delta t - 1)F_n + (-e^{c\Delta t} + c\Delta t + 1)F_{n-1}\}/(c^2\Delta t) \vdots$$

By setting s=2 we get the fourth-order exponential time differencing scheme

$$u_{n+1} = u_n e^{c\Delta t} + (\Theta_1 F_n - \Theta_2 F_{n-1} + \Theta_3 F_{n-2} \Theta_4 F_{n-3})/(6c^4\Delta t^3)$$

$$\Theta_1 = (6c^3\ \Delta t^3 + 11c^2\Delta t^2 + 12c\Delta t + 6)e^{c\Delta t} - 24c^3\ \Delta t^3 - 2611c^2\Delta t^2 - 18c\Delta t - 6$$

$$\Theta_2 = (18c^2\Delta t^2 + 30c\Delta t + 18)e^{c\Delta t} - 36c^3\ \Delta t^3 - 57c^2\Delta t^2 - 48c\Delta t - 18$$

$$\Theta_3 = (6c^2\Delta t^2 + 24c\Delta t + 18)e^{c\Delta t} - 24c^3\ \Delta t^3 - 42c^2\Delta t^2 - 42c\Delta t - 18$$

$$\Theta_4 = (2c^2\Delta t^2 + 6c\Delta t + 6)e^{c\Delta t} - 6c^3\ \Delta t^3 - 11c^2\Delta t^2 - 12c\Delta t - 6 .$$

V. On the Stability of Etdrk4 Method

The stability analysis of the ETDRK4 method is as follows (see [21,24] or[12]). For the nonlinear ODE

$$\frac{du(t)}{dt} = cu(t) + F(u(t),t) \qquad (3.1)$$

with $F(u(t),t)$ the nonlinear part, we suppose that there exists a fixed point u_0 this means that $cu_0 + F(u_0,t) = 0$. Linearizing about this fixed point, if $u(t)$ is the perturbation of u_0 and $\delta = F'(u_0,t)$ then

$$\frac{du(t)}{dt} = cu(t) + \delta u(t) \qquad (3.2)$$

and the fixed point $u_0(t)$ is stable if $\mathrm{Re}(c+\delta) < 0$.

The application of the ETDRK4 method to (3.2) leads to a recurrence relation involving u_n and u_{n+1}. Introducing the previous notation $x = \delta h$ and $y = ch$, and using the Mathematica© algebra package, we obtain the following amplification factor

$$\frac{u_{n+1}}{u_n} = r(x,y) = \ell_0 + \ell_1 x + \ell_2 x^2 + \ell_3 x^3 + \ell_4 x^4 \qquad (3.3)$$

where

$$\ell_0 = e^y$$

$$\ell_1 = \frac{-4}{y^3} + \frac{8e^{\frac{y}{2}}}{y^3} - \frac{8e^{\frac{3y}{2}}}{y^3} + \frac{4e^{2y}}{y^3} - \frac{1}{y^2} + \frac{4e^{\frac{y}{2}}}{y^2} - \frac{6e^{y}}{y^2} + \frac{4e^{\frac{3y}{2}}}{y^2} - \frac{e^{2y}}{y^2}$$

$$\ell_2 = \frac{-8}{y^4} + \frac{16e^{\frac{y}{2}}}{y^4} - \frac{16e^{\frac{3y}{2}}}{y^4} + \frac{8e^{2y}}{y^4} - \frac{5}{y^3} + \frac{12e^{\frac{y}{2}}}{y^3} - \frac{10e^{y}}{y^3} + \frac{4e^{\frac{3y}{2}}}{y^3} - \frac{e^{2y}}{y^3} - \frac{1}{y^2} + \frac{4e^{\frac{y}{2}}}{y^2} - \frac{3e^{y}}{y^3},$$

$$\ell_3 = \frac{4}{y^5} - \frac{16e^{\frac{y}{2}}}{y^5} + \frac{16e^{y}}{y^5} + \frac{8e^{\frac{3y}{2}}}{y^5} - \frac{20e^{2y}}{y^5} + \frac{8e^{5\frac{y}{2}}}{y^5} - \frac{2}{y^4} + \frac{-10e^{\frac{y}{2}}}{y^4} + \frac{16e^{y}}{y^4} - \frac{12e^{\frac{3y}{2}}}{y^4} + \frac{6e^{2y}}{y^4} - \frac{2e^{5\frac{y}{2}}}{y^4}$$
$$- \frac{2e^{\frac{y}{2}}}{y^3} + \frac{4e^{y}}{y^3} - \frac{2e^{3\frac{y}{2}}}{y^3}$$

$$\ell_4 = \frac{8}{y^6} - \frac{24e^{\frac{y}{2}}}{y^6} + \frac{16e^{y}}{y^6} + \frac{16e^{\frac{3y}{2}}}{y^5} - \frac{24e^{2y}}{y^6} + \frac{8e^{5\frac{y}{2}}}{y^6} + \frac{6}{y^5} + \frac{-18e^{\frac{y}{2}}}{y^5} + \frac{20e^{y}}{y^5} - \frac{12e^{\frac{3y}{2}}}{y^5} + \frac{6e^{2y}}{y^5} + \frac{2e^{5\frac{y}{2}}}{y^5}$$
$$+ \frac{2}{y^4} - \frac{6e^{\frac{y}{2}}}{y^4} + \frac{6e^{y}}{y^4} - \frac{2e^{3\frac{y}{2}}}{y^4}$$

An important remark: computing $\ell_0, \ell_1, \ell_2, \ell_3, \ell_4$ by the above expressions suffers from numerical instability for y close to zero. Because of that, for small y, instead of them, we will use their asymptotic expansions.

$$\ell_1 = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{13}{320}y^4 + \frac{7}{960}y^5 + O(y^6)$$

$$\ell_2 = \frac{1}{2} + \frac{1}{2}y + \frac{1}{4}y^2 + \frac{247}{2880}y^3 + \frac{131}{5760}y^4 + \frac{479}{96768}y^5 + O(y^6)$$

$$\ell_3 = \frac{1}{6} + \frac{1}{6}y + \frac{61}{720}y^2 + \frac{1}{36}y^3 + \frac{1441}{241920}y^4 + \frac{67}{120960}y^5 + O(y^6)$$

$$\ell_4 = \frac{1}{24} + \frac{1}{32}y + \frac{7}{640}y^2 + \frac{19}{11520}y^3 - \frac{311}{64512}y^4 - \frac{479}{860160}y^5 + O(y^6)$$

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We make two observations:

- As $y \mapsto 0$ y, our approximation becomes

$$r(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

which is the stability function for all the 4-stage Runge–Kutta methods of order four.

- Because c and δ may be complex, the stability region of the ETDRK4 method is four-dimensional and therefore quite difficult to represent. Unfortunately, we do not know any expression for $|r(x, y)| = 1$ we will only be able to plot it. The most common idea is to study it for each particular case; for example, assuming c to be fixed and real in [21] or that both c and δ are pure imaginary numbers in [24].

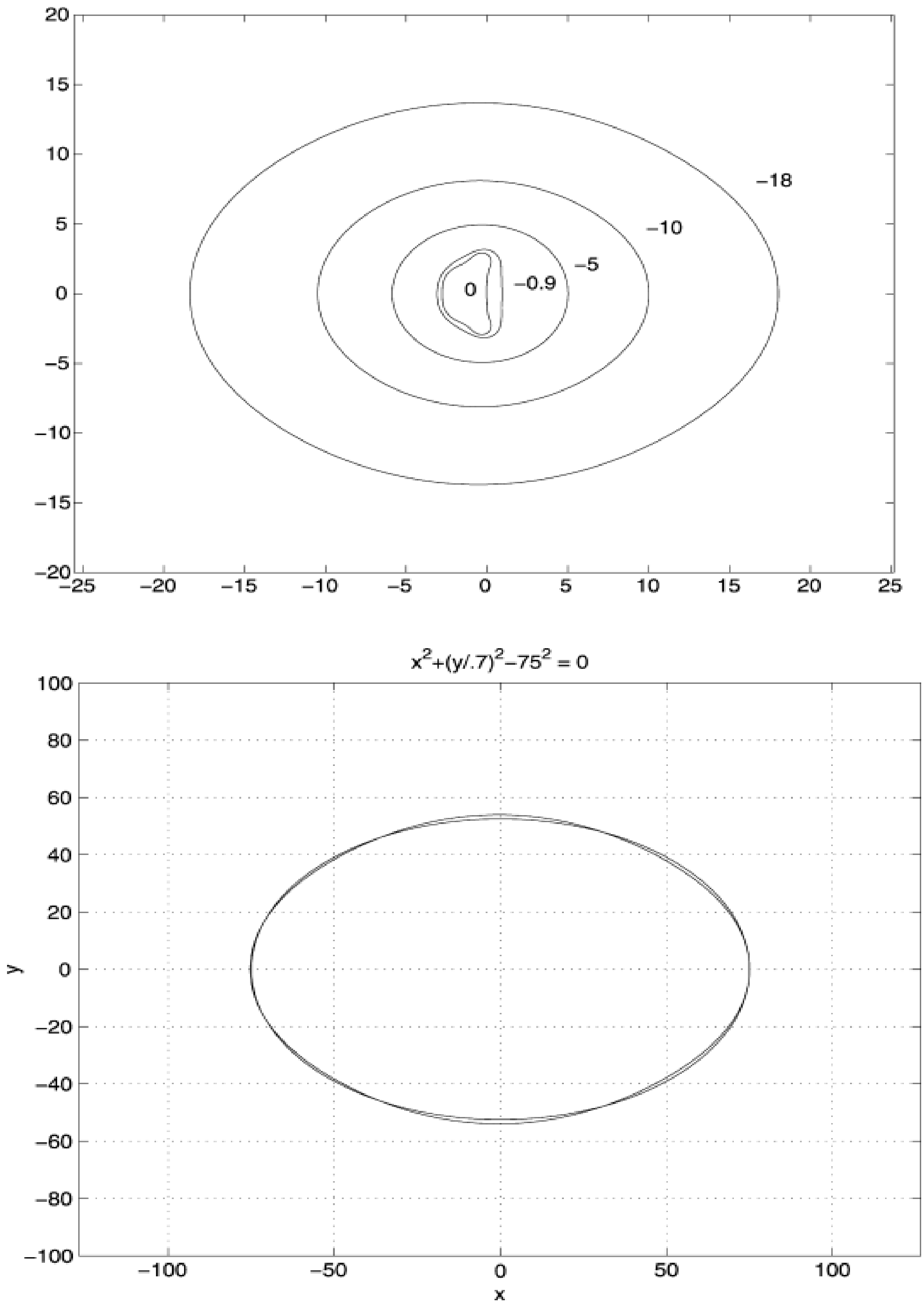

Figure 2 : Experimental boundaries and ellipse for y= −75

For dissipative PDEs with periodic boundary conditions, the scalars c that arise with a Fourier spectral method are negative. Let us take for example Burger's equation

$$u_t = \varepsilon u_{xx} - (\frac{1}{2}u^2)_x \quad x \in [-\pi, \pi] \text{ where } 0 < \varepsilon \ll 1 \quad (3.4)$$

Transforming it to the Fourier space gives

$$\tilde{u}_t = -\varepsilon\zeta^2\tilde{u} - \frac{i\zeta}{2}\tilde{u}^2 \quad \forall\zeta \qquad (3.5)$$

where $\forall\zeta$ is the Fourier wave-number and the coefficients $c = -\varepsilon\zeta^2 < 0$, span over a wide range of values when all the Fourier modes are considered. For high values of $|\zeta|$ the solutions are attracted to the slow manifold quickly because $c < 0$ and $|c| << 1$.

In *Figure. 1* we draw the boundary stability regions in the complex plane x for $y=0,-0.9,-5,-10,-18$. When the linear part is zero ($y = 0$), we recognized the stability region of the fourth-order Runge–Kutta methods and, as $y \mapsto -\infty$, the region grows. Of course, these regions only give an indication of the stability of the method.

In fact, for $y < 0$, $|y| << 1$ the boundaries that are observed approach to ellipses whose parameters have been fitted numerically with the following result.

$$(\mathrm{Re}(x))^2 + \left(\frac{\mathrm{Im}(x)}{0.7}\right)^2 = y^2 \qquad (3.6)$$

In *Figure. 2* we draw the experimental boundaries and the ellipses (3.6) with $y = -75$. The spectrum of the linear operator increases as ζ^2, while the eigenvalues of the linearization of the nonlinear part lay on the imaginary axis and increase as ζ. On the other hand, according to (3.6), when $\mathrm{Re}(x) = 0$, the intersection with the imaginary axis $\mathrm{Im}(x)$ increases as $|y|$, i.e., as ζ^2.Since the boundary of stability grows faster than , the ETDRK4 method should have a very good behavior to solve Burger's equation, which confirms the results of paper [6].

VI. Exponential Time Differencing Runge-Kutta Methods

Generally, for the one-step time-discretization methods and the Runge-Kutta methods, all the information required to start the integration is available.However, for the multi-step time-discretization methods this is not true.These methods require the evaluations of a certain number of starting values of the nonlinear term $F(u(t),t)$ at the n-th and previous time steps to build the history required for the calculations.Therefore, it is desirable to construct exponential time differencing methods that are based on Runge-Kutta methods.

Based in [12] and [3] ,Putting s=1 in equation (e_5) to get

$$a_n = u_n e^{c\Delta t} + (e^{c\Delta t} - 1)F_n / c \quad (e_6)$$

The term a_n approximates the value of u at $t_n + \Delta t$

The next step is to approximate F in the interval $t_n \le t \le t_{n+1}$ with

$$F = F_n + (t - t_n)(F(a_n, t_n + \Delta t) - F_n)/\Delta t + O(\Delta t^2) \quad (e_7)$$

and substitute into (e_1) yield

$$u_{n+1} = a_n + (e^{c\Delta t} - c\Delta t - 1)(F(a_n, t_n + \Delta t) - F_n)/(c^2\Delta t) \quad (e_8)$$

Equation (e_8) represent the first-order Runge Kutta exponential time differencing scheme

In a similar way, we can also form the second-order Runge Kutta exponential time differencing scheme

$$a_n = u_n e^{c\Delta t/2} + (e^{c\Delta t/2} - 1)F_n / c \quad (e_9)$$

As we can see equation (e_9) is formed by taking half a step of (e_6)

The next step is to approximate F in the interval $t_n \le t \le t_{n+1}$ with

$$F = F_n + \frac{(t - t_n)}{\Delta t/2}(F(a_n, t_n + \Delta t/2) - F_n) + O(\Delta t^2) \quad (e_{10})$$

By substituting (e_9) into (e_{10}) we get

$$u_{n+1} = u_n e^{c\Delta t} + \{((c\Delta t - 2)e^{c\Delta t} + c\Delta t + 2) + 2(e^{c\Delta t} - c\Delta t - 1)F(F(a_n, t_n + \Delta t/2)\}/(c^2\Delta t) \quad (e_{11}):$$

By setting s=4 an fourth-order Runge Kutta exponential time differencing scheme is obtained as follows

$$a_n = u_n e^{c\Delta t/2} + (e^{c\Delta t/2} - 1)F_n / c$$

$$b_n = u_n e^{c\Delta t/2} + (e^{c\Delta t/2} - 1)F(a_n, t_n + \Delta t/2)_n / c$$

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$$c_n = a_n e^{c\Delta t/2} + (e^{c\Delta t/2} - 1)(2F\ (b_n, t_n + \Delta t/2)\ - F_n)/c$$

$$u_{n+1} = u_n e^{c\Delta t} + \{(c^2\Delta t^2 - 3c\Delta t + 4)e^{c\Delta t} - c\Delta t - 4)F_n + 2((c\Delta t - 2)e^{c\Delta t} + c\Delta t + 2)(F(a_n, t_n + \Delta t/2) + F(b_n, t_n + \Delta t/2) \quad (e_{12})$$

$$((-c\Delta t + 4)e^{c\Delta t} - c^2\Delta t^2 + -3c\Delta t - 4)F(c_n, t_n + \Delta t)\}/(c^2\Delta t)$$

In general, the exponential time differencing Runge-Kutta method (e_{12}) has classical order four, but Hochbruck and Ostermann[11] showed that this method suffers from an order reduction. They also presented numerical experiments which show that the order reduction, predicted by their theory, may in fact arise in practical examples. In the worst case, this leads to an order reduction to order three for the Cox and Matthews method (e_{12}) [12]. However, for certain problems, such as the numerical experiments conducted by Kassam and Trefethen[13],[6] for solving various one-dimensional diffusion-type problems, and the numerical results obtained in for solving some dissipative and dispersive PDEs, the fourth-order convergence of the fourth-order Runge Kutta exponential time differencing method [12] is confirmed numerically.

Finally, we note that as $c \to 0$ in the coefficients of the s-order exponential time differencing Runge-Kutta methods, the methods reduce to the corresponding order of the Runge-Kutta schemes.

V. The Kuramoto-Sivashinsky Equation

The Kuramoto-Sivashinsky equation,is one of the simplest PDEs capable of describing complex behavior in both time and space. This equation has been of mathematical interest because of its rich dynamical properties. In physical terms, this equation describes reaction diffusion problems, and the dynamics of viscous-fuid films flowing along walls.

Kuramoto-Sivashinsky equation in one space dimension can be written

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^4 u(x,t)}{\partial x^4} \quad (14)$$

Equation (14) can be written in integral form if we introduce

$$u(x,t) = \frac{\partial \zeta(x,t)}{\partial t}$$

then

$$\frac{\partial \zeta(x,t)}{\partial t} = -\frac{1}{2}\left(\frac{\partial \zeta(x,t)}{\partial x}\right)^2 - \frac{\partial^2 \zeta(x,t)}{\partial x^2} - \frac{\partial^4 \zeta(x,t)}{\partial x^4} \quad (15)$$

or in form

$$u_t + \nabla^4 u + \nabla^2 u + |\nabla\ u|^2/2 = 0$$

The Kuramoto-Sivashinsky equation with $2L$ periodic boundary conditions in Fourier space can be written as follows

$$u_j(x) = \sum_{k=0}^{N_\tau - 1} \tilde{u}_k\ e^{\frac{ik\pi x}{L}} \quad (16)$$

$$u_j(x,t) = \sum_{k=0}^{N_\tau - 1} \tilde{u}_k\, e^{\frac{ik\pi x}{L}} \quad (16.1)$$

$$\frac{\partial u_j(x,t)}{\partial t} = \sum_{k=0}^{N_\tau - 1} \frac{d\tilde{u}_k}{dt} \tilde{u}_k\, e^{\frac{ik\pi x}{L}} \quad (16.2)$$

$$\frac{\partial u_j(x,t)}{\partial x} = \sum_{k=0}^{N_\tau - 1} \frac{ik\pi}{L} \tilde{u}_k\, e^{\frac{ik\pi x}{L}} \quad (16.3)$$

$$\frac{\partial^2 u_j(x,t)}{\partial x^2} = \sum_{k=0}^{N_\tau - 1} \left(\frac{ik\pi}{L}\right)^2 \tilde{u}_k\, e^{\frac{ik\pi x}{L}} \quad (16.4)$$

$$\frac{\partial^4 u_j(x,t)}{\partial x^4} = \sum_{k=0}^{N_\tau - 1} \left(\frac{ik\pi}{L}\right)^4 \tilde{u}_k\, e^{\frac{ik\pi x}{L}} \quad (16.5)$$

If we substitute into (14) we get

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^4 u(x,t)}{\partial x^4} \Leftrightarrow \frac{\partial u_j(x,t)}{\partial t} = -u_j(x,t)\frac{\partial u_j(x,t)}{\partial x} - \frac{\partial^2 u_j(x,t)}{\partial x^2} - \frac{\partial^4 u_j(x,t)}{\partial x^4}$$

$$\Leftrightarrow \sum_{k=0}^{N_\tau - 1} \frac{d\tilde{u}_k}{dt} e^{\frac{ik\pi x}{L}} = -\left[\left(\sum_{k=0}^{N_\tau - 1} \tilde{u}_k\, e^{\frac{ik\pi x}{L}}\right) * \left(\sum_{k=0}^{N_\tau - 1} \frac{ik\pi}{L} \tilde{u}_k\, e^{\frac{ik\pi x}{L}}\right)\right] - \left[\sum_{k=0}^{N_\tau - 1} \left(\frac{ik\pi}{L}\right)^2 \tilde{u}_k\, e^{\frac{ik\pi x}{L}}\right] - \left[\sum_{k=0}^{N_\tau - 1} \left(\frac{ik\pi}{L}\right)^4 \tilde{u}_k\, e^{\frac{ik\pi x}{L}}\right]$$

By simplifying (16), (16.1), (16.2), (16.3), (16.4), (16.5) and note that

$$\tilde{u}_k = \frac{1}{N_\tau}\sum_{j=0}^{N_\tau-1} u_j e^{-2ijk/N_\tau}, \quad (i)^2 = -1, (i)^4 = 1 \quad x_j = jh, h = \frac{2L}{N_\tau}$$

Equation (14) can be written as fellows

$$\frac{\partial \tilde{u}_k(t)}{\partial t} = \left(k^2 - k^4\right)\tilde{u}_k(t) - \frac{ik}{2}\varpi_k \tag{17}$$

Where $\varpi_k = \frac{1}{2L}\int_{-L}^{L} u^2(x,t) e^{\frac{ik\pi x}{L}} dx = \frac{1}{N_\tau}\sum_{j=0}^{N_\tau-1} u^2{}_j(x,t) e^{-2ijk/N_\tau} = FFT\left[u(t)^2\right]$

In final form will be

$$\frac{\partial \tilde{u}_k(t)}{\partial t} = \left(k^2 - k^4\right)\tilde{u}_k(t) - \frac{ik}{2}FFT\left[u(t)^2\right] \tag{18}$$

Equation has strong dissipative dynamics, which arise from the fourth order dissipation $\frac{\partial^4 u(x,t)}{\partial x^4}$ term that provides damping at small scales. Also, it includes the mechanisms of a linear negative diffusion $\frac{\partial^2 u(x,t)}{\partial x^2}$ term, which is responsible for an instability of modes with large wavelength,i.e small wave-numbers. The nonlinear advection/steepening $u(x,t)\frac{\partial u(x,t)}{\partial x}$ term in the equation transforms energy between large and small scales.

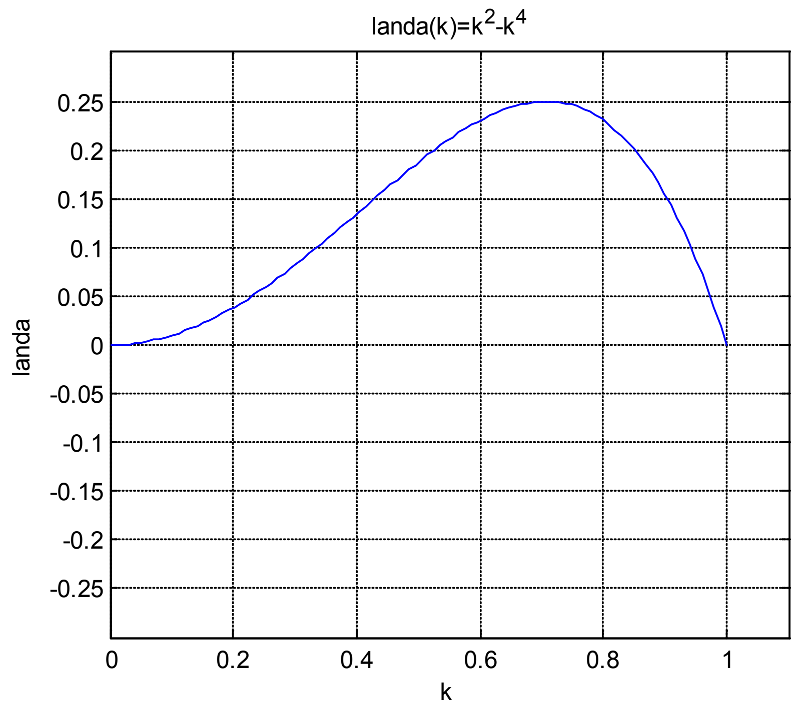

Figure 3 : The growth rate $\lambda(k)$ for perturbations of the form $e^{\lambda t}e^{ikx}$ to the zero solution of the Kuramoto-Sivashinsky (K-S) equation

The zero solution of the K-S equation is linearly unstable (the growth rate $\lambda(k) > 0$ for perturbations of the form $e^{\lambda t}e^{ikx}$ to modes with wave-numbers $|k| = \left|\frac{2\pi}{\vartheta}\right| < 1$ for a wavelength ϑ and is damped for modes with $|k| > 1$ see *Figure. 3* these modes are coupled to each other through the non-linear term.

The stiffness in the system (17) is due to the fact that the diagonal linear operator $\left(k^2 - k^4\right)$, with the elements, has some large negative real eigenvalues that represent decay, because of the strong dissipation, on a time scale much shorter than that typical of the nonlinear term. The nature of the solutions to the the Kuramoto-Sivashinsky equation varies with the system size of linear operator. For large size of linear operator, enough unstable Fourier modes exist to make the system chaotic. For small size of linear operator, insuffcient Fourier modes exist, causing the system to approach a steady state solution. In this case, the exponential time differencing methods integrate the system very much more accurately than other methods since the the exponential time differencing methods assume in their derivation that the solution varies slowly in time.

VI. Numerical Result

For the simulation tests, we choose two periodic initial conditions

$$u_1(x) = e^{\cos\left(\frac{x}{2}\right)}, x \in [0, 4\pi]$$

$$u_2(x) = 1.7\cos\left(\frac{x}{2}\right) + 0.1\sin\left(\frac{x}{2}\right) + 0.6\cos(x) + 2.4\sin(x), x \in [0, 4\pi]$$

When evaluating the coefficients of the exponential time differencing and the exponential time differencing Runge Kutta methods via the "Cauchy integral" approach [5],[6] we choose circular contours of radius R = 1. Each contour is centered at one of the elements that are on the diagonal matrix of the linear part of the semi-discretized model. We integrate the system (17) using fourth-order Runge Kutta exponential time differencing scheme using $N_\tau = 64$ with time-step size $\Delta t = 2e-10$.

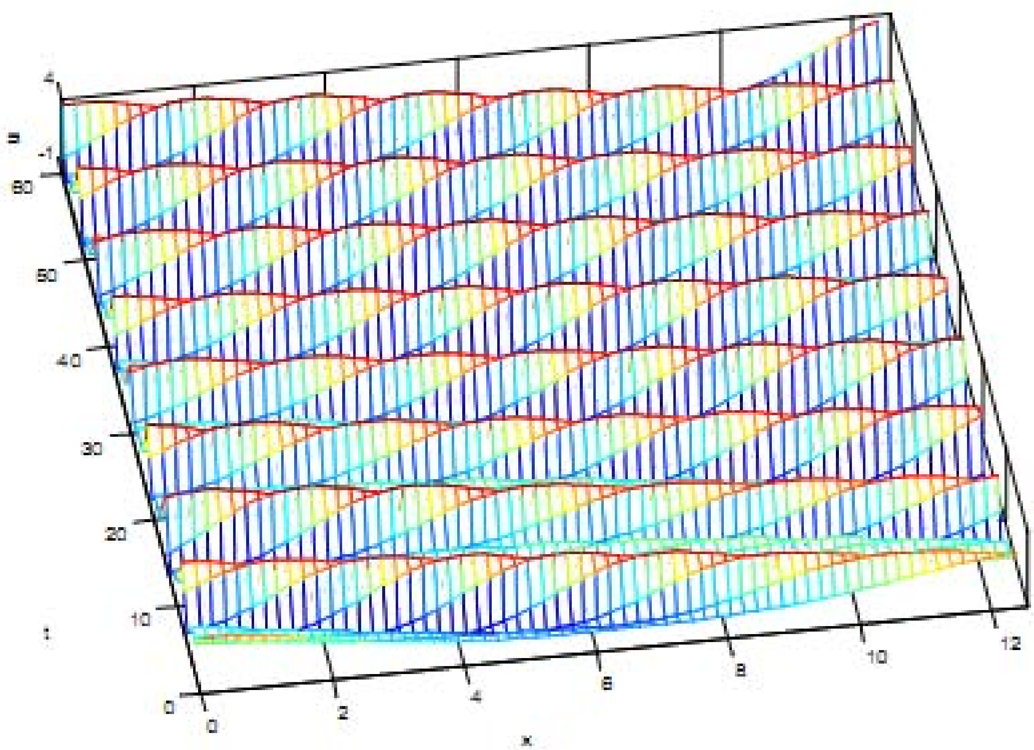

Figure 1 : Time evolution of the numerical solution of the Kuramoto-Sivashinsky up to t = 60 with the initial condition $u_1(x) = e^{\cos\left(\frac{x}{2}\right)}, x \in [0, 4\pi]$

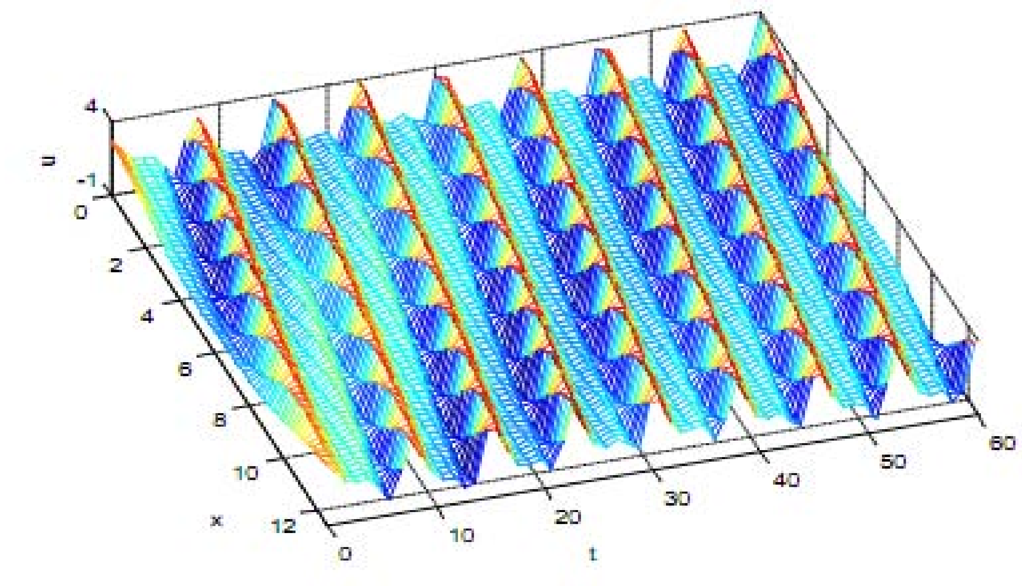

Figure 2 : Time evolution of the numerical solution of the Kuramoto-Sivashinsky up to t = 60 with the initial condition $u_1(x) = e^{\cos\left(\frac{x}{2}\right)}, x \in [0,4\pi]$

The solution, in the figure 1 with the initial condition $u_1(x) = e^{\cos\left(\frac{x}{2}\right)}, x \in [0,4\pi]$ with $N_\tau = 64$ and time-step size $\Delta t = 2e-10$, appears as a mesh plot and shows waves propagating, traveling periodically in time and persisting without change of shap.

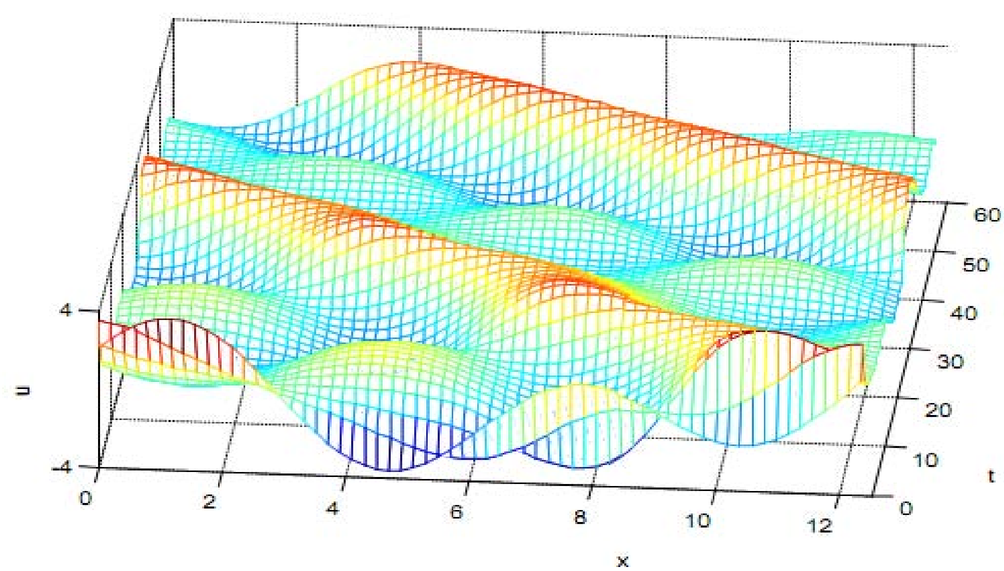

Figure 3 : Time evolution of the numerical solution of the Kuramoto-Sivashinsky up to t = 60 with the initial condition $u_2(x) = 1.7\cos\left(\frac{x}{2}\right) + 0.1\sin\left(\frac{x}{2}\right) + 0.6\cos(x) + 2.4\sin(x), x \in [0,4\pi]$

In the figure 2 with the initial condition $u_2(x) = 1.7\cos\left(\frac{x}{2}\right) + 0.1\sin\left(\frac{x}{2}\right) + 0.6\cos(x) + 2.4\sin(x), x \in [0,4\pi]$ with $N_\tau = 64$ and time-step size $\Delta t = 2e-10$, the solution appears as a mesh plot and shows waves propagating, traveling periodically in time and persisting without change of shape.

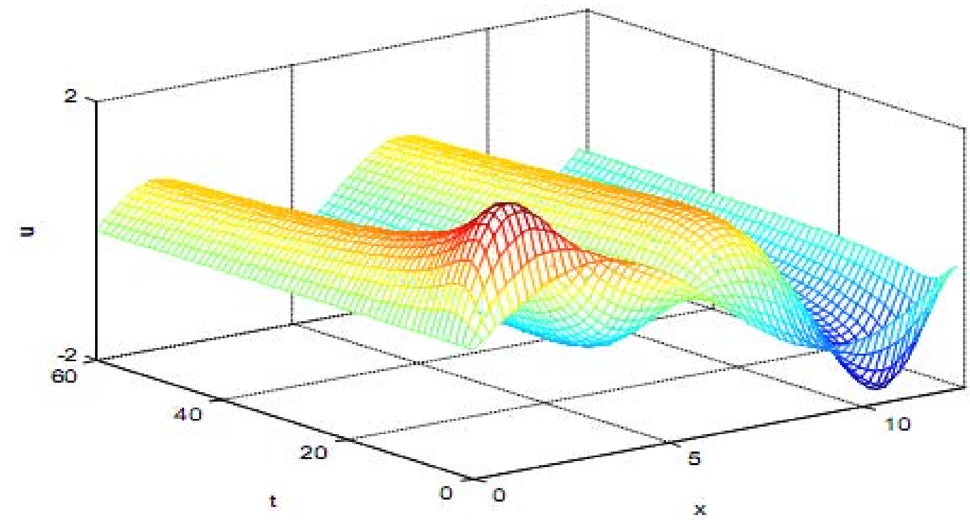

Figure 4 : Time evolution of the numerical solution of the Kuramoto-Sivashinsky up to t = 60 with the initial condition $u_1(x) = \sin(x/2), x \in [0,4\pi]$

In the figure 3 with the initial condition $u_1(x) = \sin(x/2), x \in [0,4\pi]$ with $N_\tau = 64$ and time-step size $\Delta t = 2e-10$, the solution appears more clear as a mesh plot and shows waves propagating, traveling periodically in time and persisting without change of shape.

VII. Conclusions

In this paper, the main objective of this study was for finding the solution of one dimensional semilinear fourth order hyperbolic *Kuramoto-Sivashinsky* equation, describing reaction diffusion problems, and the dynamics of viscous-fuid films flowing along walls. In order to achieve this, we applied Fourier spectral approximation for the spatial discretization. In addition, we evaluated the coeffcients of the exponential time differencing and the exponential time differencing –fourth order Runge Kutta methods via the "Cauchy integral" .Some typical examples have been demonstrated in order to illustrate the efficiency and accuracy of the exponential time differencing methods technique in this case. For the simulation tests, we chose periodic boundary conditions and applied Fourier spectral approximation for the spatial discretization. In addition, we evaluated the coefficients of the Exponential Time Differencing Runge-Kutta methods via the "Cauchy integral" approach. The equations can be used repeatedly with necessary adaptations of the initial conditions.

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