

# A Probabilistic Approach to Portfolio Optimization under Epistemic Uncertainty

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*Received: 8 December 2013 Accepted: 2 January 2014 Published: 15 January 2014*

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## Abstract

Portfolio is an appropriate collection or list of investments held by an organization or an individual where Portfolio optimization means to calculate the optimal weights for the collection or list of investments that gives the highest return with least risk. In previous works on this topic did not consider correlation factor and epistemic uncertainty on returns. In our model both the factors are considered because correlation between the assets exists in real life and epistemic uncertainty may exist in different rate of return of different assets. In this work two models are developed to optimize a portfolio. Two models are Optimization model without EU and Optimization model with EU. By optimization we can decide the assets on which we can invest, how much to invest and can take divestment decisions.

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**Index terms**— portfolio epistemic uncertainty (EU) covariance correlation.

## 1 Introduction

Portfolio is an appropriate collection or list of investments held by an organization or an individual. Portfolio optimization means to calculate the optimal weights for the collection or list of investments that gives the highest return with least risk. In this work we have used a probabilistic approach to optimize the portfolio. Existing models on this topic by other authors [e.g., 1] did not consider correlation factor and epistemic uncertainty on returns. In our model both the factors are considered. Epistemic uncertainty is the situation where we do not do things correctly. Epistemic uncertainty results in data because of not measuring a quantity with sufficient accuracy, or because of building a model by neglecting certain effects, or because of hidden data. Moreover in "portfolio optimization (2 Assets)" by Markowitz [1] proportion of investment was calculated with considering a riskless asset and two risky assets. In another paper [2] optimization of investments were calculated by the linear combination of the assets. This means in that model the co-variances of assets were not considered. Covariance measures the specified value by which two random variables change together. In our model co-variances between assets are considered. Kawas and Thiele [3] developed Log robust model for both the independent and correlated assets to optimize the portfolio. The continuous probability distribution of a random variable whose logarithm is normally distributed is called Log normal distribution [4]. They had considered the uncertainty on the rate of return. But Log robust model incorporates the randomness in the rate of returns of the assets. In our model no trends in the rate of return is required. Buckley et al [5] formulated the optimization of the investment where the rates of return of the assets have Gaussian mixture distribution. The continuous probability distribution that has a bell shaped probability density function is known as Gaussian distribution [6]. In our model we have considered that the rate of return or the range of rate of return will be known.

## 2 II.

Epistemic Uncertainty (eu), Correlation and Covariance

### 3 a) Epistemic Uncertainty (EU)

Epistemic uncertainty is the situation where we do not do things correctly. Epistemic uncertainty results in data because of not measuring a quantity with sufficient accuracy, or because of building a model by neglecting certain effects, or because of hidden data [7]. In our model epistemic uncertainty exists in the rate of returns of assets. We do not know what will be the future rate of return of an asset. Moreover, we have used the rate of return on particular time interval where the rate of return is changing continuously. Therefore, by considering the epistemic uncertainty we will know the range of rate of return for an asset rather than the particular rate of return.

### 4 b) Correlation

One of the most commonly used statistics is correlation. It measures and describes the degree of relationship between two variables. To compute the correlation of two variables "Pearson's product-moment" method can be used. "Pearson's product-moment" correlation is computed by dividing the covariance with standard deviation [8].

### 5 Global Journal of Researches in

Engineering ( ) G Volume XIV Issue III Version I 1 Year 2014 So,  $\text{corr}(x,y) = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y}$ ,  $\text{cov}(x,y) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$

Here,

Correlation is +1 in the case of a positive or increasing linear relationship, -1 in the case of a decreasing or negative linear relationship and some value between -1 and 1 in all other cases, indicating the degree of linear dependence between the variables. As it approaches zero, it implies that there is no relationship. The closer the coefficient is to either -1 or 1, the stronger the correlation between the variables.

### 6 c) Covariance

Covariance measures the specified value by which two random variables change together. If larger values of one variable correspond to the larger values of the other variable and the same holds for the smaller values, the covariance is positive. In the opposite case, when the larger values of one variable correspond to the smaller values of the other, the covariance is negative. Therefore, the sign of the covariance shows the trend in the linear relationship between the variables. Covariance can be calculated as follows [9]:

### 7 Model Structure

Optimization refers to the selection of a best element from some set of available alternatives. Portfolio is an appropriate collection or list of investments held by an organization or an individual. Portfolio optimization means to calculate the optimal weights for the collection or list of investments that gives the highest return with least risk. We have developed two models for portfolio optimization: 1. Optimization Model without EU

### 8 Optimization Model with EU a) Optimization Model without EU

Let us assume that a set of  $n$  assets and a sum of money are given, which to be invested to maximize the total return on investment. We have to find the investment proportions  $x_i$  ( $i = 1, 2, \dots, n$ ) to be invested in assets  $i = 1, 2, \dots, n$  so that the total return on investment over a period is maximized. It is not possible to determine with certainty because the actual returns of the assets are not known. However, if we assume that the expected returns of the assets and their variances and co-variances are known, we can calculate the investment proportions that results in high expected return with low variance. As the model does not consider epistemic uncertainty, we know the specified value of the returns. In this model we will maximize the total return while simultaneously minimizing the risk.

$$9 \quad \text{Cov}(x,y) = E[(x-E[x])(y-E[y])]$$

(2)

$$10 \quad \text{Cov}(x,y) = E[xy] - E[x]E[y]$$

(

Here,  $\text{Cov}(x,y)$  = Covariance of  $x$  and  $y$ ,  $E[x]$  = Mean of  $x$ ,  $E[y]$  = Mean of  $y$ ,  $E[xy]$  = Mean of  $xy$ ,  $n$  = Number of available assets,  $r_i$  = Expected return of assets,  $x_i$  = Fraction of the capital invested in asset  $i$ ,  $i, j = 1, 2, \dots, n$

The covariance of the return for assets  $i$  and  $j$  is  $\text{Cov}(x_i, x_j) = \frac{1}{n} \sum_{k=1}^n (x_{ik} - \bar{x}_i)(x_{jk} - \bar{x}_j)$  (4) Subject to,  $\sum_{i=1}^n x_i = 1$  (6)

where  $i$  and  $j = 1, 2, \dots, n$  set of  $n$  assets and a sum of money are given, which are to be invested to maximize the total return on investment. As the model considers the epistemic uncertainty, we know the specified value of the returns of some assets and a range of the returns for the rest number of assets. Let a set of

p assets whose specified return values are known and a set of q assets whose range of returns are known [where  $p+q = n$ ]. We have to find the investment proportions  $x_i$  to be invested in assets  $i = 1, 2 \dots n$  so that the total return on investment over a period is maximized. Therefore, we consider additional constraints in this model. The solution is obtained by simultaneously solving two objective functions. We have to find the proportion of investments in one function, the returns and co-variance of asset set p in another function. As for the asset set q, the ranges of returns are known; we can get the range of covariances for asset set q. Let,  $x_i$  = Fraction of the capital invested in asset i  $i, j =$

The covariance of the returns of assets i and j lb = lower bound of returns for asset set q ub = upper bound of returns for asset set q Therefore, the objective function 1 becomes, 
$$\max_{x} \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij} \quad (8)$$
 Subject to, 
$$f_2(x) = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij} \quad (9)$$

$\text{? i x} = 1 \text{ (10)x i 0}$

where  $i$  and  $j = 1, 2, \dots, n$ . And the objective function 2 becomes,  $\max_{i,j} f(x) = \sum_{i,j} c_{ij} x_{ij}$  (12) Subject to,  $f_2(x) = \sum_{i,j} c_{ij} x_{ij} \leq b_k$  (13)  $\min_{i,j} f_3(x) = \sum_{i,j} c_{ij} x_{ij}$  (14)  $\max_{i,j} f_4(x) = \sum_{i,j} c_{ij} x_{ij}$  (15)

where  $k$  is the asset identity of asset set  $q$ .

(11) For the asset set  $q$  described by epistemic uncertainty, the additional constraints in Eqs. (??4 ? = 21 ? = 493.07; 11 ? = 2906.9; 22 ? = 583.79.

Optimized portfolio:

By using (4), (??), (6) and (??) the equations become: Under this model we will consider the epistemic uncertainty. So we know the specified value of the returns of some assets and a range of the returns for the rest number of assets.  $\text{Max } f_1(x) =$

## 11 Example Problem 2

Let, we have three assets (asset 1, asset 2 and asset 3). The rate of return of asset 1 and 2 are known. But we know the limit of the rate of return of asset 3.

12 Table 3 :

The rates of returns of asset 1 and asset 2 Table ?? : The limits of asset 3

In the following discussion, we solve the above problem using our proposed model that considers the epistemic uncertainty.

## 13 Covariance

If we want to optimize the assets portfolio then we have to determine six co variances. Average rate of return of asset 2 ( $r_2$ ) = 47.28 But we will have a limit for the rate of return of asset 3.

Lower limit of the average rate of return of asset 3 (lb) =  $(62+60+61+59+58)/5 = 60$

Upper limit of the average rate of return of asset 3(ub) = (71+72+75+73+70)/5 = 72.2 1 ? 11 2 ? 12 = ? 21 3 ? 13 = ? 31 4 ? 225 ? 23 = ? 32 6 ? 33

We can easily calculate  $\pi_{11}$ ,  $\pi_{22}$  and  $\pi_{12} = \pi_{21}$  because we know the mean rate of return of asset 1 and 2. But we can know the lower (minimum value) and the upper (maximum value) limit of  $\pi_{33}$ ,  $\pi_{13} = \pi_{31}$  and  $\pi_{23} = \pi_{32}$  because we know the limit for asset 3.  $\pi_{33}$  can easily be calculated by using the MATLAB code.

By solving, the limit for  $\gamma = 33 = [4 \times 10^{-10}, 245]$ . Optimized portfolio:

We will construct two objective functions to optimize the returns. By using Eqs. (??), (9), (??0) and (11) we get,  $\max_{x_1, x_2, x_3} f = w^*(r_1 x_1 + r_2 x_2 + r_3 x_3) - v^*(x_1^2 + x_2^2 + x_3^2 + 2x_1 x_2 + 2x_1 x_3 + 2x_2 x_3)$  (16) Subject to,  $w^*(r_1 x_1 + r_2 x_2 + r_3 x_3) - v^*(x_1^2 + x_2^2 + x_3^2 + 2x_1 x_2 + 2x_1 x_3 + 2x_2 x_3) \geq 994.26$ ;

$$x_1 + x_2 + x_3 = 1;$$
$$x_1, x_2, x_3 \geq 0.$$

By using  $w = .5$  and  $v = .5$  with assumed value of  $x_1$ ,  $x_2$  To solve  $?_{13} = ?_{31}$  and  $?_{23} = ?_{32}$  we have to know the maximum and minimum value of  $E(?_{13})$

and E(23).

V.

## 14 Findings

To calculate the optimal weights for the collection or list of investments that gives the highest return with least risk is called portfolio optimization. In this work we have used a probabilistic approach to optimize the portfolio. We have developed two models to optimize the asset portfolio. The two models optimize the portfolio in different approach. One considers the epistemic uncertainty and the other does not consider the epistemic uncertainty. By using these models we can optimize the portfolio of different investments, different shares. If we look through the results then we can see how the results define how much to invest in which asset.

Firstly, the two models optimize the portfolio by considering not only the rate of returns but also the risks. We have considered the risk portion in the constraint area from where we can minimize it and simultaneously we have considered it in the objective function also.

Secondly, we have not only calculated the range of co variances but also optimized the values of the co variances. The risks become less while the co variances become less.

Thirdly, we have shown the examples of the models by using two or three assets. However, the models can optimize as many assets as are there. But in that case we have to calculate the co variances with proper situation. That means we have to calculate the co variances by considering whether there exists epistemic uncertainty or not.

Fourthly, in the second model "Optimization Model with EU" we have considered the epistemic uncertainty. We have optimized the proportions of investments with simultaneously optimizing the co variances and the rate of returns for the assets under uncertain returns. We could optimize the co variances and rate of returns for the assets under uncertain returns by considering a separate objective functions and not considering the proportions of assets. In that case the calculation will be simpler than the present calculation but we will not have a properly optimized solution.

Finally, in optimizing the portfolio we have considered two notations  $w$  and  $v$ , where they are weighting coefficients that represent the relative importance of the objective functions. That means  $w$  and  $v$  represent the relative importance of the total return and the risk respectively. If we have a portfolio where the risk part has less importance then we can use a lower  $v$  value than the  $w$  value. And if we have a portfolio where the risk part has a greater importance then we can use a  $v$  value with comparison to the  $w$  value.

So the models optimize a portfolio regardless the number of assets by both considering and not considering the epistemic uncertainty, giving relative importance factor to the total return part and the risk part.

VI.

## 15 Conclusion

Optimization refers to the selection of a best element from some set of available alternatives. Portfolio is an appropriate collection or list of investments held by an organization or an individual. Portfolio optimization means to calculate the optimal weights for the collection or list of investments that gives the highest return with least risk. In this work we have developed two models using probabilistic approach to optimize the portfolio. As we have stated earlier the findings of my thesis that the models optimize a portfolio regardless of the number of assets by both considering and not considering the epistemic uncertainty, giving importance factor to the total return part and the risk part. By using these models we can optimize the portfolio of investments, shares etc. We can know the proportions of investment on particular assets in efficient way that minimizes the risk on profitability. However, the models use discrete time interval values of the rate of returns. Moreover, the code used for the model "Optimization Model with EU" is not more flexible. Further research can be done by developing the model for the continuous values of the rate of return and developing the code for the model "Optimization Model with EU" by incorporating more flexibility. <sup>1</sup>

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Figure 1: 2 A

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Global Journal of Researches in Engineering

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Figure 2: Table 1 :

Here,

$w$  and  $v$  are the weighting coefficients that represent the relative importance of the total return and the risk respectively, where  $w+v = 1$

If  $S_{\min}$  and  $S_{\max}$  are the minimum and maximum attainable values then  $[S_{\min}, S_{\max}]$

We can calculate the co variances of the assets by Equ (3).

| YEAR | 2NDICB |
|------|--------|
| 1    | 89.32  |
| 2    | 70.75  |
| 3    | 41.06  |
| 4    | 48.86  |
| 5    | 78.77  |
| 6    | 80.75  |
| 7    | 66.06  |
| 8    | 76.15  |
| 9    | 107.14 |
| 10   | 135.55 |

Average rate of return:

Here,  $r_1 = 79.441$  and

$r_2 = 63.819$  (mean values)

Covariance:

As we know,  $\sigma_{ij} = E(xy) - E(x) \cdot E(y)$

Here,  $E(x) = 79.441$ ;  $E(y) = 63.819$  and  $E(xy) = 5562.914$

So,  $\sigma_{12}$

Figure 3:

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Year 2014

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Figure 4: Table 5 :

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| w                         | v  | x 1                       | x 2         | Optimized return                                 |
|---------------------------|----|---------------------------|-------------|--------------------------------------------------|
| 1                         | 0  | .5                        | .5          | -71.63                                           |
| .1                        | .9 | .0366                     | .9634       | 516.0148                                         |
| .2                        | .8 | .0370                     | .9630       | 451.5250                                         |
| .3                        | .7 | .0376                     | .9624       | 387.0342                                         |
| .4                        | .6 | .0383                     | .9617       | 322.5419                                         |
| .5                        | .5 | .0393                     | .9607       | 258.0474                                         |
| .6                        | .4 | .0409                     | .9591       | 193.5487                                         |
| .7                        | .3 | .0435                     | .9565       | 129.0420                                         |
| .8                        | .2 | .0487                     | .9513       | 64.5149                                          |
| .9                        | .1 | .0643                     | .9359       | -.0933                                           |
| 0                         | 1  | .0362                     | .9638       | 580.5039                                         |
|                           |    |                           | Lower limit | Upper limit                                      |
|                           |    |                           | 62          | 71                                               |
|                           |    |                           | 60          | 72                                               |
|                           |    |                           | 61          | 75                                               |
|                           |    |                           | 59          | 73                                               |
|                           |    |                           | 58          | 70                                               |
| Rate of return of asset 1 |    | Rate of return of asset 2 |             |                                                  |
| 89.32                     |    | 45.01                     |             |                                                  |
| 70.75                     |    | 54.05                     |             | Average rate of return:                          |
| 41.06 48.86               |    | 53.41                     |             | Average rate of return of asset 1 (r 1 ) = 65.75 |
|                           |    | 42.74                     |             |                                                  |
| 78.77                     |    | 41.21                     |             |                                                  |

Figure 5: Table 2 :

|             |                                                                                                   |
|-------------|---------------------------------------------------------------------------------------------------|
| I           |                                                                                                   |
| ( ) G       |                                                                                                   |
| Volume      |                                                                                                   |
| XIV Issue   |                                                                                                   |
| III Version |                                                                                                   |
| Global      | By using Eqs. (12), (13), (14) and (15) we get, 31 23 33 3 1 , , , ) ( min ? ? ? r x f = w*(r 1 * |
| Journal of  |                                                                                                   |
| Researches  |                                                                                                   |
| in Engi-    |                                                                                                   |
| neering     |                                                                                                   |
|             | w*(r 1 *x 1 +r 2 *x 2 +r 3 *x 3 ) -                                                               |
|             | v*(x 1                                                                                            |
|             | 2 2 *? 22 +x                                                                                      |
|             | *? 3 2 *? 33                                                                                      |
|             | 11 +2*x 1 *x 2                                                                                    |
|             | +x *? 12 +2*x                                                                                     |
|             | 2 2 *x 3 *? 23                                                                                    |
|             | +2*x 3 *x 1                                                                                       |
|             | *? 31 ) ?                                                                                         |
|             | 994.26;                                                                                           |

Figure 6:



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