

Correlation and Distribution Analyses of Estimated Fractal Dimensions and Hurst's Exponent from Waveforms of Excited Nonlinear Pendulum

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Abstract

This study utilised correlation and distribution analyses to investigate the acceptability of application of two fractal dimension estimators to characterise the waveforms originating from excited nonlinear pendulum. Parameters selection sensitive simulation of the excited nonlinear pendulum waveforms was performed with the constant time step fourth order Runge-Kutta algorithm with codes developed in FORTRAN90. However, the waveforms validated by Gregory and Jerry (1990) and treated as time series were characterized using developed codes of Carlos (1998) and Hurst fractal dimension estimation procedures. The validation results compare qualitatively well and the correlation coefficients between Carlos (1998)-based and Hurst's exponent based dimension estimate for the angular displacement and velocity are respectively $R^2 = 0.68$ and $R^2 = 0.66$. A higher correlation coefficient ($R^2 = 0.84$) existed between the estimated Hurst's exponent of the angular displacement and velocity. The Hurst distribution exhibited both full spectrum and peak values range 0.04 to 1.00 and percentage probability range 2 to 12. The sum of this study results is the interchange possibility and utility of the two fractal dimension estimators as waveforms characterising tool.) existed between the estimated Hurst's exponent of the angular displacement and velocity. The Hurst distribution exhibited both full spectrum and peak values range 0.04 to 1.00 and percentage probability range 2 to 12. The sum of this study results is the interchange possibility and utility of the two fractal dimension estimators as waveforms characterising tool.

Index terms— fractal dimension, hurst's exponent, excited nonlinear pendulum, waveforms and runge-kutta.

1 Introduction

electronic circuit modelling of a ball bouncing on an oscillating table in demonstrating these different dynamic characteristics. Findings show that the fractal dimension of 1.07 and 1.7 respectively implies regular data and chaotic data. It was also shown that the fractal dimension which is infinite refers to data that are random. It is concluded from the paper that the bouncing ball circuit system dynamics has a chaotic attractor of low fractal dimension. This paper has shown that fractal dimension can be utilised in characterising nonlinear system. Salau and Ajide (2012) asserted that the study of images play a significant role in engineering and several fields of study. The authors employed fractal dimension as the estimator for sectional images characterisation of selected dynamic systems. Findings obtained from the paper showed the high potentiality of fractal disk dimension as characterising tool for images.

40 The chaotic driven impact of two important parameters of excited Duffing oscillator has been studied using
 41 fractal disk dimension (Salau and Ajide, 2013). The outcome of the study has provided a very robust platform for
 42 the relevance of the use of fractal dimension as a reliable estimator for characterising nonlinear dynamic systems.

43 Hurst exponent can easily be computed from fractal dimension. Wikipedia (2013) describes Hurst exponent
 44 as what can be utilized as a measure of long term memory of time series. It relates to the autocorrelations
 45 of the time series and the rate at which these decrease as the lag between pairs of values increases. It was
 46 further understood here that Hurst exponent were originally developed in hydrology for the practical matter
 47 of determining optimum dam sizing for the Nile River's volatile rain and drought conditions that have been
 48 observed over a long period of time. The name "Hurst Exponent" was coined from Harold Edwin Hurst who
 49 was the lead researcher in this field. An extract from Ian (2013) article shows that Hurst exponent has a very
 50 wide application. It occurs in several areas of mathematics (Fractals and Chaos theory, Spectral Analysis, e.t.c.),
 51 biophysics, computer networking, hydrology and just to mention but a few. Hurst exponent has been found to
 52 be very useful in marketing, medicine, engineering and host of other fields. In business, the simulation model
 53 which describes the Idrich et al in 2001 describes fractals as a rough shape that can easily be subdivided in parts
 54 of which are at least approximately a reduced copy of the whole (self-similar). The use of a concept known as
 55 fractal dimension has made it easy for researcher in this field to measure the extent to which a onedimensional
 56 thread fills up a three dimensional space (Scott,1990). The use of fractal dimension as an estimating tool for
 57 characterising nonlinear systems is becoming more appreciated in the recent times. The A multi-level supply chain
 58 has been done using Hurst exponent (Chien-Yuan and Jinsheng, 2013). The exponent satisfactorily analyzes the
 59 dynamic behaviour of inventory under various factors that include lead time, demand pattern, information sharing
 60 and RFID (Radio Frequency Identification). The outcome of the study showed that the lead time and RFID
 61 utilization effectualness greatly influence the inventory dynamics under the stock and supply line discrepancies
 62 of specified parameters. This paper has also shown the utility of Hurst exponent as a relevant estimator of
 63 supply chain dynamics. The trend in prices of gold has been analyzed by Priyadarshini and Babu (2010). The
 64 author utilised fractal dimension index (FDI) that was computed from Hurst exponents. The monthly gold rates
 65 required for study was collected from January 1971 to April 2010 (40 years). The results of FDI calculated
 66 from a constant Hurst exponent ($H=1.047$) revealed that it is useful tool for determining the amount of market
 67 growth. The author affirmed that this study has opened-up an immeasurable advantage for a businessman to
 68 venture into markets that have the most opportunity. Yu-Zhi et al (2011) paper dwell on the possibility of Hurst
 69 exponent being utilised in ecology. A careful application of this exponent to rodent populations revealed that it
 70 is very convenient and effective exponent for detecting nonlinear systems in natural populations. In medicine,
 71 Kamalanand and Jawahar (2012) paper studied the behaviour of Human Immuno Virus (HIV) system using the
 72 three dimensional HIV model and a chaotic measure popularly referred to as Hurst exponent. Results of the
 73 study showed that Hurst exponents of cells and viral load vary nonlinearly in the selected parameters range. It
 74 was further shown that the three dimension HIV model can accommodate both persistent and anti-persistent
 75 dynamics of HIV states. The research output of this paper has clearly shown the high clinical relevance of Hurst
 76 exponent. This is because the analysis of the complexity of the HIV model is helpful for choosing appropriate
 77 parameter estimation methods. It has great benefits for identifying suitable treatment strategies.

78 Despite the richness of fractal dimensions and Hurst exponent as estimating tools in nonlinear dynamics, they
 79 are yet to be extensively explored in many nonlinear mechanical engineering system dynamics as characterising
 80 exponents. Extensive literature study shows that Duffing oscillator and excited nonlinear pendulum are just two
 81 of the so many mechanical systems that Hurst exponent has not been significantly used to characterize. The main
 82 objective of this paper is to investigate the correlation and distribution analyses of estimated Fractal dimensions
 83 and Hurst's exponent from Waveforms of Excited Nonlinear Pendulum. The quest of filling this research gaps is
 84 a strong motivation for this paper.

2 II.

3 Methodology a) Equation of Motion

87 According to Gregory and Jerry (1990), the dimensionless representation of the damped, sinusoidally excited
 88 pendulum with fixed lumped-mass and length is described by equation (1) called equation of motion. This
 89 equation expresses Newton's second law with the various terms on the left representing respectively variation with
 90 time (t) of the acceleration, damping and gravitation effect. The variation with time of the angular displacement,
 91 velocity and acceleration are $(\theta)''$, $(\theta)'$ and (θ) . The angular velocity of the forcing is $D\theta$, g is
 92 the forcing amplitude (not gravitational acceleration) and q is the damping parameter.) to a pair of first order
 93 differential equation leads to equations (2) and (3).

$$(\theta)'' + q(\theta)' + D\theta = g \cos(\theta) \sin(\theta) \quad (3)$$

94
 95 The transient and steady angular displacements and velocities were obtained by simultaneous simulation of
 96 equations (3) and (4) using constant time step Runge-Kutta fourth order algorithms over large number of
 97 excitation periods. The results of the periodic time history of the steady angular displacement and velocity are
 98 the waveforms objects for the present investigation.

4 b) Fractal Dimension of Waveforms

This study utilised Carlos (1998) procedure to evaluate the fractal dimension of the waveforms. The method is of Hausdorff dimension (where for a curve: D_H) by equations (1) to (7). The transformation maps N -points of angular displacement and velocity waveforms to another that belongs to a unit square respectively. In the resulting unit square, Carlos (1998) justified that each of the waveforms can be visualised as covered by a grid of $N \times N$ cells where N of them containing one point of the transformed waveforms. Substituting in equation (4) the length (L) of the transformed waveform and N where $N = \dots$

results in a modified expression for dimension given by equation (8). $\lim_{N \rightarrow \infty} \frac{\ln(L)}{\ln(N)} = D_H$ (4)

* $\min_{i=1, \dots, N} \max_{j=1, \dots, N} \ln(L) - \ln(N) = D_H$ (8)

It is to be noted that the approximation to D_H expressed in equation (8), improves as $N \rightarrow \infty$.

5 c) Hurst Exponent (H) of Waveforms

According to Mandelbrot (1983), Hurst (1951), Aniel and Benjamin (2005) the Hurst exponent H can be obtained from the rescaled range (R/S) statistic which is the range (R) of partial sums of deviation of times series from its mean, rescaled by its standard deviation (S). In the present study, the rescaled range (R/S) and the standard deviation (S) for the periodic time history of the steady displacement and velocity waveforms are given correspondingly by equations (9 & 11) and (10 & 12) for the time span (T). Literatures recommended that time span (T) to ensure the reliability of the estimated Hurst exponent, which is the slope of best line to the log-log plot of (R/S) versus (T) .

$\frac{1}{T} \max_{k=1, \dots, T} (S_k) - \min_{k=1, \dots, T} (S_k) = R$ (9)

$\frac{1}{T} \max_{k=1, \dots, T} (S_k) - \min_{k=1, \dots, T} (S_k) = R$ (10)

6 ()

$k k S \dots = \dots$ (11)

7 ()

$k k S \dots = \dots$ (12)

In a related study on fractal dimensions of time sequences by Sy-Sang and Feng-Yuan (2009) equation (13) give the expression for the relationship between the dimension and the Hurst exponent. $D_H = \dots$ (13)

It is important to note that in the present study the estimated fractal dimension using equations (8) and (13) are respectively represented by DCS-Carlos (1998) based and DHE-Hurst's exponent based.

8 d) Studied Cases and Simulation Parameters

The present investigation focuses three cases at fixed excitation frequency (ω), the details are provided as follow:

Case-I : ($\omega = 2, 0.9 q g = \dots$). The case enables the investigation of the variation of estimated fractal dimension with increasing number of time series to verify existence of dimension convergence.

Case-II : ($\omega = 4, 1.5 q g = \dots$). The original and the transformed Poincare sections obtained from this case was compared visually with its equivalent from Gregory and Jerry (1990) Case-III : This case focuses the parameter plane defined by $2.0 \leq \omega \leq 4.0$ and $0.9 \leq q g \leq 1.5$.

. First part of the case dealt with a total of 2601 nodal points selected on the plane including 51 coordinates each along the axes Simulation of equations (1) and (3) for each set of parameters selected with fourth order Runge-Kutta algorithms were effected from initial conditions (0, 0) using constant time step (Comparison of the distribution of the estimated Hurst exponents was investigated in 100-equal intervals between the limits-values (Hurst exponents) for the angular displacement and velocity.

9 III.

10 Results and Discussion

Figure 1 refers. The visual quality of the untransformed and the transformed Poincare sections are same and compare very well with the corresponding result reported by Gregory and Jerry (1990). This demonstrated topological invariance under linear transformation as required by Carlos (1998) approach of estimating fractal dimension. Likewise the invariance of Poincare section (periodic phase plots) suggested the invariance of the associated angular displacement and velocity waveforms that are the objects of analyses in this study. However, quantitative comparison of the limits-values along the angular displacement (AD) and angular velocity (AV) axes are different. Higher correlation coefficients may be achieved by increasing the number of time series beyond its present value i.e. 3000-for dimension estimate and 2048-for Hurst exponent estimate. The quality of the visual pattern created by the scatter plots of the dimensions correlation of angular displacement (AD) and angular

154 velocity (AV) are the same. It is important to note that the small and negligible quantitative difference between
 155 the obtained correlation coefficients is an indication of common source of time series (i.e. the source is periodic
 156 sampling of solutions to harmonically excited nonlinear pendulum). In figure 5, the normalised probability
 157 distribution of the estimated Hurst's exponents of the angular displacement waveforms peaked at three different
 158 Hurst-values: 1.00, 0.53 and 0.20. The corresponding percentage probability is 9, 5 and 3. Likewise the distinct
 159 distribution peak points for the angular velocity waveforms are four at Hurst-values: 0.99, 0.50, 0.20 and 0.04.
 160 The corresponding percentage probability is 9, 3, 5 and 2. The full spectrum of Hurst-values (0 1 H ? ?)
 161 coupled with the multiple peak points observed in these distributions is an indication of the richness of the
 162 pendulum dynamics when driven by arbitrary parameters selection from this parameter plane while holding
 163 constant the drive frequency. That is the pendulum can behave periodically, quasi-periodically or worst still
 164 chaotically depending on the choices of parameters from this plane.

165 IV.

166 11 Conclusions

167 This study established good correlation coefficients for two differently estimated fractal dimensions while the
 168 distribution of Hurst exponents exhibited full spectrum for the waveforms from excited nonlinear pendulum.
 169 The Hurst exponents' distributions apart from having full spectrum show multiple distinct peak values as a
 170 possible indicator of pendulum multiple behaviours such as periodic, quasi-periodic, random or chaotic. The
 171 study therefore show that the pendulum drive parameters plane consist of large selection combinations with
 172 associated rich dynamics. Thus the present study has established the possible interchange and utility of the two
 fractal dimension estimators as waveforms characterising tool. ^{1 2 3 4}



Figure 1:

173

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²1, 1, 1, 1

³2, 2, 2, 1

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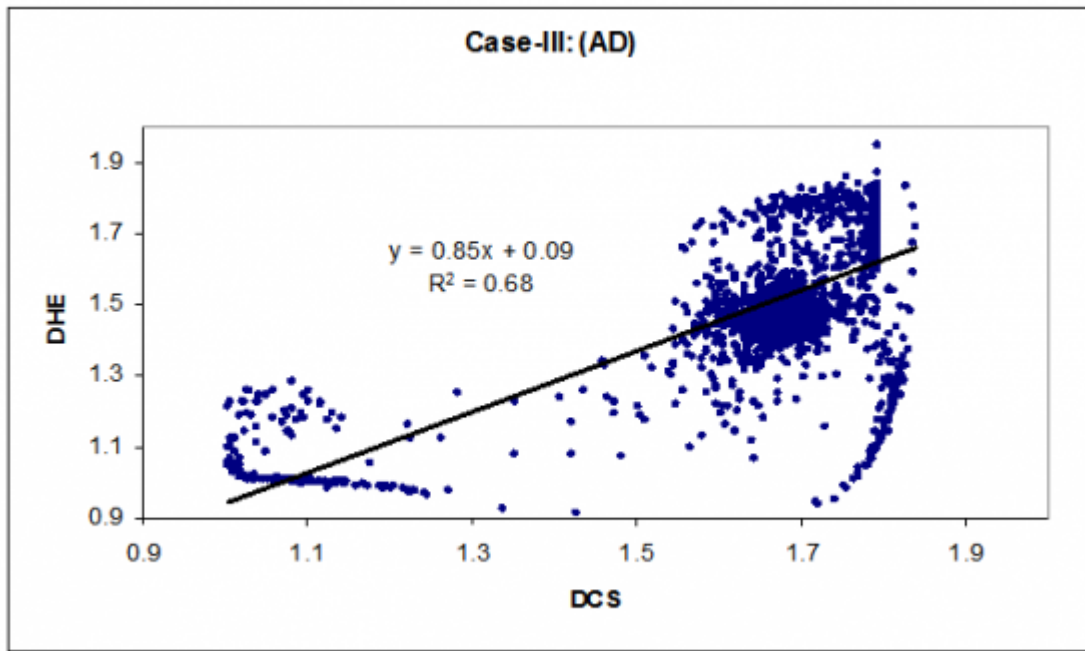


Figure 2:

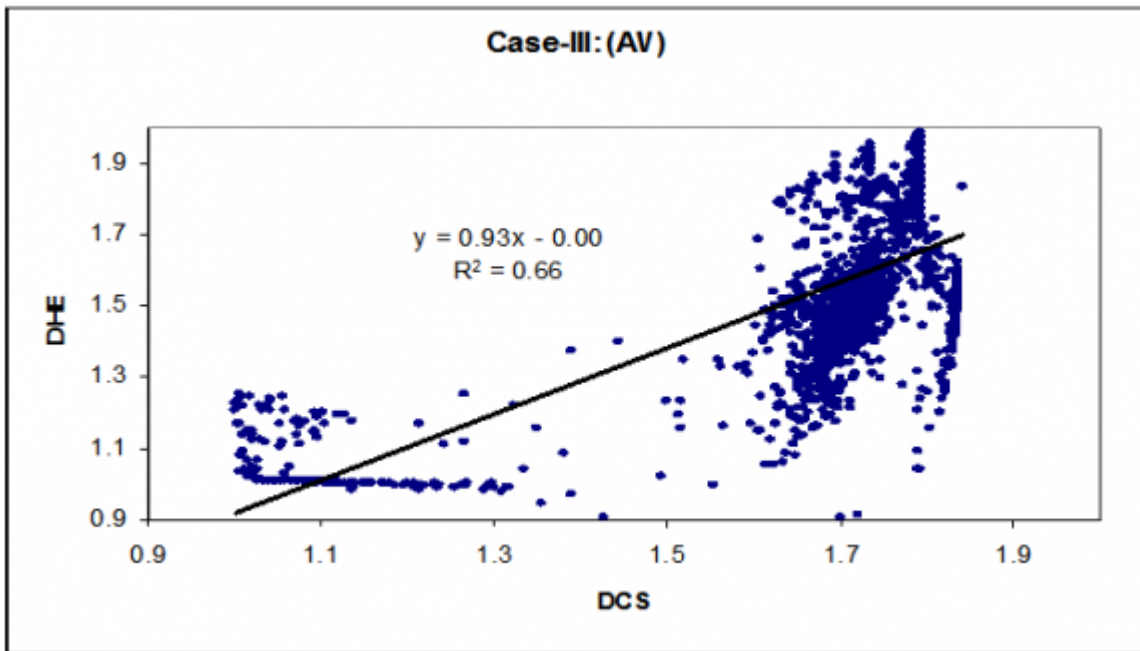


Figure 3:

are respectively $R = 0.68$ and $R = 0.66$. A higher correlation coefficient ($R = 0.84$)

[Note: Salau T. A.O. & Ajide O.O.]

Figure 4:

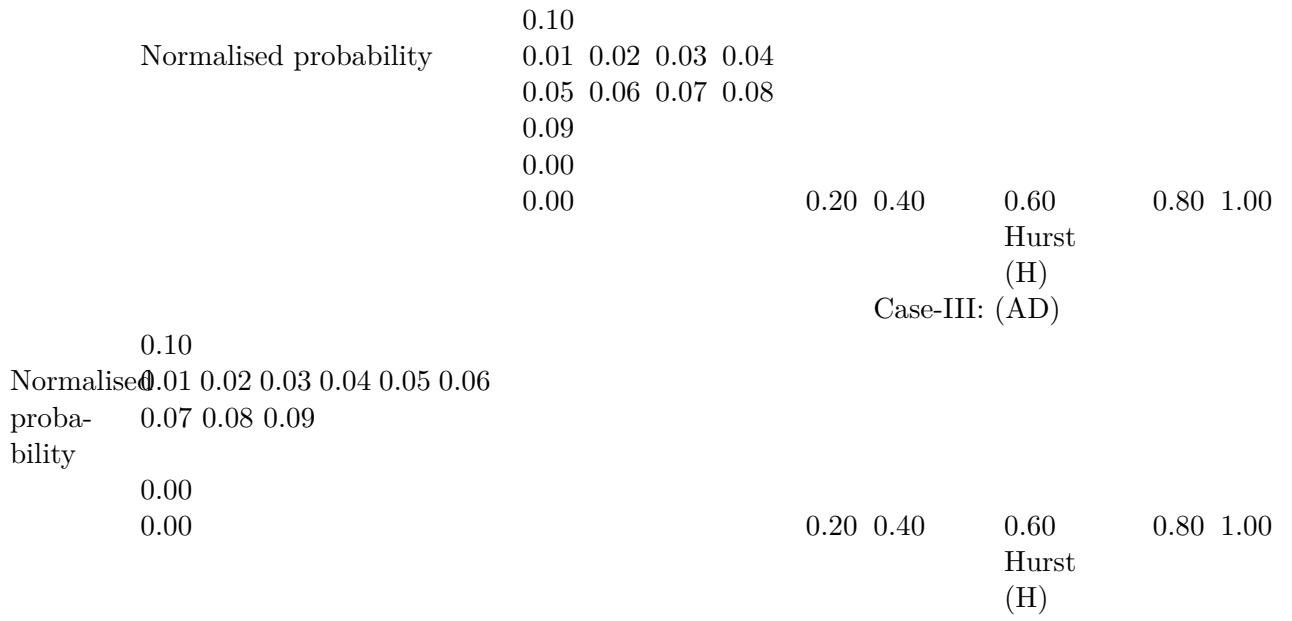


Figure 5:

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